Devil Round

1. Find all x such that $(\ln(x^4))^2 = (\ln(x))^6$.

2. On a piece of paper, Alan has written a number N between 0 and 2010, inclusive. Yiwen attempts to guess it in the following manner: she can send Alan a positive number M, which Alan will attempt to subtract from his own number, which we will call N. If M is less than or equal N, then he will erase N and replace it with $N - M$. Otherwise, Alan will tell Yiwen that $M > N$. What is the minimum number of attempts that Yiwen must make in order to determine uniquely what number Alan started with?

3. How many positive integers between 1 and 50 have at least 4 distinct positive integer divisors? (Remember that both 1 and n are divisors of n.)

4. Let F_n denote the n^{th} Fibonacci number, with $F_0 = 0$ and $F_1 = 1$. Find the last digit of

$$\sum_{i=0}^{97!+4} F_i.$$

5. Find all prime numbers p such that $2p + 1$ is a perfect cube.

6. What is the maximum number of knights that can be placed on a 9×9 chessboard such that no two knights attack each other?

7. S is a set of 9 consecutive positive integers such that the sum of the squares of the 5 smallest integers in the set is the sum of the squares of the remaining 4. What is the sum of all 9 integers?

8. In the following infinite array, each row is an arithmetic sequence, and each column is a geometric sequence. Find the sum of the infinite sequence of entries along the main diagonal.

$$\begin{array}{ccccc} 1 & ? & \frac{1}{2} & ? & \cdots \\ \frac{1}{2} & ? & ? & ? & \cdots \\ ? & ? & ? & ? & \cdots \\ ? & ? & ? & ? & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

9. Let $x > y > 0$ be real numbers. Find the minimum value of $\frac{x}{y} + \frac{4x}{x-y}$.

10. A regular pentagon $P = A_1A_2A_3A_4A_5$ and a square $S = B_1B_2B_3B_4$ are both inscribed in the unit circle. For a given pentagon P and square S, let $f(P, S)$ be the minimum length of the minor arcs A_iB_j, for $1 \le i \le 5$ and $1 \le j \le 4$. Find the maximum of $f(P, S)$ over all pairs of shapes.

11. Find the sum of the largest and smallest prime factors of $9^4 + 3^4 + 1$.

12. A transmitter is sending a message consisting of 4 binary digits (either ones or zeros) to a receiver. Unfortunately, the transmitter makes errors: for each digit in the message, the probability that the transmitter sends the correct digit to the receiver is only 80%. (Errors are independent across all digits.) To avoid errors, the receiver only accepts a message if the sum of the first three digits equals the last digit modulo 2. If the receiver accepts a message, what is the probability that the message was correct?

13. Find the integer N such that

$$\prod_{i=0}^{8} \sec\left(\frac{\pi}{9} 2^i\right) = N.$$