Duke Math Meet 2012

Individual Round

1. Vivek has three letters to send out. Unfortunately, he forgets which letter is which after sealing the envelopes and before putting on the addresses. He puts the addresses on at random sends out the letters anyways. What are the chances that none of the three recipients get their intended letter?

2. David is a horrible bowler. Luckily, Logan and Christy let him use bumpers. The bowling lane is 2 meters wide, and David's ball travels a total distance of 24 meters. How many times did David's bowling ball hit the bumpers, if he threw it from the middle of the lane at a 60° degree angle to the horizontal?

3. Find $\gcd(212106, 106212)$.

4. Michael has two fair dice, one six-sided (with sides marked 1 through 6) and one eight-sided (with sides marked 1-8). Michael play a game with Alex: Alex calls out a number, and then Michael rolls the dice. If the sum of the dice is equal to Alex's number, Michael gives Alex the amount of the sum. Otherwise Alex wins nothing. What number should Alex call to maximize his expected gain of money?

5. Suppose that x is a real number with $\log_5 \sin x + \log_5 \cos x = -1$. Find

$$|\sin^2 x \cos x + \cos^2 x \sin x|.$$

6. What is the volume of the largest sphere that fits inside a regular tetrahedron of side length 6?

7. An ant is wandering on the edges of a cube. At every second, the ant randomly chooses one of the three edges incident at one vertex and walks along that edge, arriving at the other vertex at the end of the second. What is the probability that the ant is at its starting vertex after exactly 6 seconds?

8. Determine the smallest positive integer k such that there exist m, n non-negative integers with $m > 1$ satisfying

$$k = 2^{2m+1} - n^2.$$

9. For $A, B \subset \mathbb{Z}$ with $A, B \neq \emptyset$, define $A + B = \{a + b | a \in A, b \in B\}$. Determine the least n such that there exist sets A, B with $|A| = |B| = n$ and $A + B = \{0, 1, 2, ..., 2012\}$.

10. For positive integers $n \geq 1$, let $\tau(n)$ and $\sigma(n)$ be, respectively, the number of and sum of the positive integer divisors of n (including 1 and n). For example, $\tau(1) = \sigma(1) = 1$ and $\tau(6) = 4$, $\sigma(6) = 12$. Find the number of positive integers $n \leq 100$ such that

$$\sigma(n) \leq (\sqrt{n} - 1)^2 + \tau(n)\sqrt{n}.$$