Duke Math Meet 2012

Team Round

1. Let 2^k be the largest power of 2 dividing $30! = 30 \cdot 29 \cdot 28 \cdots 2 \cdot 1$. Find k.

2. Let $d(n)$ be the total number of digits needed to write all the numbers from 1 to n in base 10; for example, $d(5) = 5$ and $d(20) = 31$. Find $d(2012)$.

3. Jim and TongTong play a game. Jim flips 10 coins and TongTong flips 11 coins; whoever gets the most heads wins. If they get the same number of heads, there is a tie. What is the probability that TongTong wins?

4. There are a certain number of potatoes in a pile. When separated into mounds of three, two remain. When divided into mounds of four, three remain. When divided into mounds of five, one remain. It is clear there are at least 150 potatoes in the pile. What is the least number of potatoes there can be in the pile?

5. Call an ordered triple of sets (A, B, C) **nice** if $|A \cap B| = |B \cap C| = |C \cap A| = 2$ and $|A \cap B \cap C| = 0$. How many ordered triples of subsets of $\{1, 2, \cdots, 9\}$ are nice? (Note: $|S|$ denotes the number of elements of S, and $S \cap T$ denotes the intersection of S and T.)

6. Brett has an $n \times n \times n$ cube (where n is an integer) which he dips into blue paint. He then cuts the cube into a bunch of $1 \times 1 \times 1$ cubes, and notices that the number of un-painted cubes (which is positive) evenly divides the number of painted cubes. What is the largest possible side length of Brett's original cube?

7. Choose two real numbers x and y uniformly at random from the interval $[0, 1]$. What is the probability that x is closer to $1/4$ than y is to $1/2$?

8. In triangle ABC, we have $\angle BAC = 20^\circ$ and $AB = AC$. D is a point on segment AB such that $AD = BC$. What is $\angle ADC$, in degrees?

9. Let a, b, c, d be real numbers such that

$$ab + c + d = 2012, bc + d + a = 2010, cd + a + b = 2013, da + b + c = 2009.$$

   Find d.

10. Let $\theta \in [0, 2\pi)$ such that $\cos\theta = 2/3$. Find

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \cos(n\theta).$$