Duke Math Meet 2013-14

Team Round

1. Suppose 5 bales of hay are weighted two at a time in all possible ways. The weights obtained are 110, 112, 113, 114, 115, 116, 117, 118, 120, 121. What is the difference between the heaviest and the lightest bale?

2. Paul and Paula are playing a game with dice. Each have an 8-sided die, and they roll at the same time. If the number is the same they continue rolling; otherwise the one who rolled a higher number wins. What is the probability that the game lasts at most 3 rounds?

3. Find the unique positive integer n such that $\frac{n^3+5}{n^2-1}$ is an integer.

4. How many numbers have 6 digits, some four of which are 2, 0, 1, 4 (not necessarily consecutive or in that order) and have the sum of their digits equal to 9?

5. The Duke School has N students, where N is at most 500. Every year the school has three sports competitions: one in basketball, one in volleyball, and one in soccer. Students may participate in all three competitions. A basketball team has 5 spots, a volleyball team has 6 spots, and a soccer team has 11 spots on the team. All students are encouraged to play, but 16 people choose not to play basketball, 9 choose not to play volleyball and 5 choose not to play soccer. Miraculously, other than that all of the students who wanted to play could be divided evenly into teams of the appropriate size. How many players are there in the school?

6. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_1 = 0$ and $a_{n+1} = \frac{a_n - \sqrt{3}}{\sqrt{3}a_n + 1}$. Find $a_1 + a_2 + \cdots + a_{2014}$.

7. A soldier is fighting a three-headed dragon. At any minute, the soldier swings her sword, at which point there are three outcomes: either the soldier misses and the dragon grows a new head, the soldier chops off one head that instantaneously regrows, or the soldier chops off two heads and none grow back. If the dragon has at least two heads, the soldier is equally likely to miss or chop off two heads. The dragon dies when it has no heads left, and it overpowers the soldier if it has at least five heads. What is the probability that the soldier wins?

8. A rook moves alternating horizontally and vertically on an infinite chessboard. The rock moves one square horizontally (in either direction) at the first move, two squares vertically at the second, three horizontally at the third and so on. Let S be the set of integers n with the property that there exists a series of moves such that after the n-th move the rock is back where it started. Find the number of elements in the set $S \cap \{1, 2, ..., 2014\}$.

9. Find the largest integer n such that the number of positive integer divisors of n (including 1 and n) is at least $\sqrt{n}$.

10. Suppose that x, y are irrational numbers such that $xy, x^2 + y, y^2 + x$ are rational numbers. Find $x + y$.