Individual Round

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Problem 1. Trung has 2 bells. One bell rings 6 times per hour and the other bell rings 10 times per hour. At the start of the hour both bells ring. After how much time will the bells ring again at the same time? Express your answer in hours.

Solution. The first bell rings at an interval of 10 minutes. The second bell rings at an interval of 6 minutes. Hence, the bells will take $\gcd(10, 6) = 30$ minutes, which is $\boxed{\frac{1}{2}}$ hour. $\square$

Problem 2. In a soccer tournament there are n teams participating. Each team plays every other team once. The matches can end in a win for one team or in a draw. If the match ends with a win, the winner gets 3 points and the loser gets 0. If the match ends in a draw, each team gets 1 point. At the end of the tournament the total number of points of all the teams is 21. Let p be the number of points of the team in the first place. Find $n + p$.

Solution. Since each team plays every other team once, there will be $\binom{n}{2} = \frac{n(n-1)}{2}$ matches. Let the number of matches that end with wins be x and end in draws be y. If the match ends with a win, the total number of points will increase by 3. If the match ends in a draw, each team gets 1 point, so the total number of points will increase by 2. Hence, we have $3x + 2y = 21$.
When $n = 4$, there are 6 games in total. Since $3 \times 6 < 21$, we have $n > 4$. When $n = 6$, there are 15 games in total. Since $2 \times 15 > 21$, we have $n < 6$. Thus, $n = 5$ and there are 10 games, so $x + y = 10$.
Therefore, we can have $x = 1$ and $y = 9$. Then, we have 3 teams tie with all others, 1 team wins one game and ties all others, and 1 team loses one game and ties all others. Therefore, the team in the first place wins 1 game and ties 3 games, so $p = 6$. Hence, $n + p = \boxed{11}$. $\square$

Problem 3. What is the largest 3 digit number $\overline{abc}$ such that

$$b \cdot \overline{ac} = c \cdot \overline{ab} + 50?$$

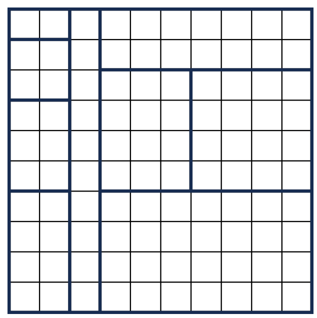
Solution. From the equation, we have $b \cdot (10a + c) = c \cdot (10a + b) + 50$, which can be simplified to $a \cdot (b - c) = 5$. Since we want $\overline{abc}$ to be the greatest, so a should be the greatest in the first place, so $a = 5$ and $b - c = 1$. Since b and c are one digits, we should take $b = 9$ and $c = 8$ to maximize $\overline{abc}$. Hence, $\overline{abc} = \boxed{598}$. $\square$

Problem 4. Let $s(n)$ be the number of quadruplets (x, y, z, t) of positive integers with the property that $n = x + y + z + t$. Find the smallest n such that $s(n) > 2014$.

Solution. Let $x' = x - 1$, $y' = y - 1$, $z' = z - 1$, and $t' = t - 1$, so $x' + y' + z' + t' = n - 4$. Since x, y, z, t must be at least 1, x', y', z', t' must be at least 0. Then, by stars and bars, we have $s(n) = \binom{(n-4)+3}{3} = \binom{n-1}{3}$ quadruplets. Since $\binom{24}{3} = 2024$, the smallest n such that $s(n) > 2014$ is $\boxed{25}$. $\square$

Problem 5. Consider a decomposition of a 10×10 chessboard into p disjoint rectangles such that each rectangle contains an integral number of squares and each rectangle contains an equal number of white squares as black squares. Furthermore, each rectangle has different number of squares inside. What is the maximum of p?

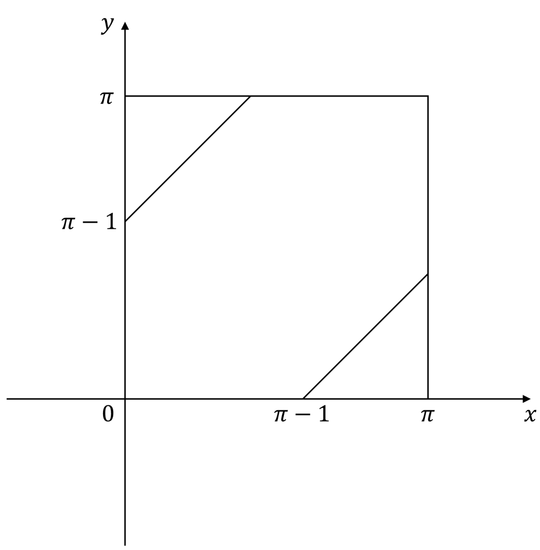
Solution. Since each rectangle contains an equal number of white squares as black squares, the rectangle contains an even number of squares. When a rectangle contains an even number of squares, the rectangle contains an equal number of white squares as black squares, since each white square is connected with black square, vice versa. Hence, each rectangle contains an equal number of white squares as black squares if and only if the number of squares is even.
Thus, since it's a 10×10 chessboard, we have $a_1 + a_2 + \cdots + a_p = 50$, with each rectangle an area of $2a_i$ for $1 \geq i \geq p$. To have the maximum of p, we should take a_i as small as possible. Since they have to be distinct, we have $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 14 = 50$.



The decomposition is possible in this case, so $p = \boxed{9}$. □

Problem 6. If two points are selected at random from a straight line segment of length π, what is the probability that the distance between them is at least $\pi - 1$?

Solution. Let the distances from the left vertex be x and y, respectively. We have $0 \leq x, y \leq \pi$ and $\pi - 1 \leq |x - y| \leq \pi$. Hence, we can have the following graph:



Hence, the probability that the distance between them is at least $\pi - 1$ is $\boxed{\frac{1}{\pi^2}}$ □

Problem 7. Find the length n of the longest possible geometric progression $a_1, a_2, \dots, a_n$ such that the a_i are distinct positive integers between 100 and 2014 inclusive.

Solution. Let the common ratio be $\frac{p}{q}$, where p, q are positive integers that are coprime. WLOG, we can assume $p > q$.
Consider the case when $q = 1$. We have the common ratio to be p. Since we want the longest possible geometric progression, p should be at the minimum, so $p = 2$. Then, $a_n = a_1 \cdot 2^{n-1}$. Since $a_1 \geq 100$ and $a_n \leq 2014$, we can have $a_1 = 100$, so the geometric progression is $\{100, 200, 400, 800, 1600\}$. Hence, $n = 5$.
Consider the case when $q = 2$. We have the common ratio to be $\frac{p}{2}$. Since we want the longest possible geometric progression, p should be at the minimum, so $p = 3$. Then, $a_n = a_1 \cdot (\frac{3}{2})^{n-1}$. Since a_n is an integer, a_1 must divide 2^{n-1}. Since $a_1 \geq 100$ and $a_n \leq 2014$, we can have $a_1 = 128$, so the geometric progression is $\{128, 192, 288, 432, 648, 972, 1459\}$. Hence, $n = 7$.
Consider the case when $q \geq 3$. We have $a_n = a_1 \cdot (\frac{p}{q})^{n-1}$. If $n > 7$, we should have a_1 to divide q^{n-1}. Since $q \geq 3$ and $n > 7$, we have $q^{n-1} \geq 2187$. Since $a_1 \leq 2014$, there is no such geometric progression.
Hence, we have $n = \boxed{7}$. □

Problem 8. Feng is standing in front of a 100 story building with two identical crystal balls. A crystal ball will break if dropped from a certain floor m of the building or higher, but it will not break if it is dropped from a floor lower than m. What is the minimum number of times Feng needs to drop a ball in order to guarantee he determined m by the time all the crystal balls break?

Solution. Let the number of times Feng needs to drop a ball be n. The first ball should be dropped at the nth floor. If it breaks, Feng can drop another ball from the 1st to the $(n-1)$th floor to guarantee he determined m, which ends up n times. If it doesn't break, Feng can drop it again at the $(2n-1)$th floor. If it breaks, Feng can drop another ball from the $(n+1)$th floor to the $(2n-2)$ th floor to guarantee he determined m, which also ends up n times.
Hence, we should have

$$\sum_{i=1}^{n} i \geq 100.$$

Since $n = 14$ is the minimum, the number of times Feng needs to drop a ball is $\boxed{14}$. □

Problem 9. Let A and B be disjoint subsets of $\{1, 2, \dots, 10\}$ such that $|A \cup B| = 10$ the product of the elements of A is equal to the sum of the elements in B. Find how many such A and B exist.

Solution. Consider the case when $|A| = 1$. Let the element in A be x. We have $x = 55 - x$. Hence, we have $2x = 55$, so there is no such A.

Consider the case when $|A| = 2$. Let the element in A be x, y. WLOG, we can assume $x > y$. We have $xy = 55 - x - y$. Hence, we have $xy + x + y + 1 = 56$, so $(x+1)(y+1) = 56$. Since x, y are integers from 1 to 10, we have $x = 7$ and $y = 6$.

Consider the case when $|A| = 3$. Let the element in A be x, y, z. WLOG, we can assume $x > y > z$. We have $xyz = 55 - x - y - z$.
When $z = 1$, we have $xy + x + y + 1 = 55$, so $(x+1)(y+1) = 55$. Since x, y are integers from 1 to 10, we have $x = 10$ and $y = 4$.
When $z = 2$, we have $2xy + x + y = 53$, so $4xy + 2x + 2y + 1 = 107$. Then, we have $(2x+1)(2y+1) = 107$. Since 107 is a prime number, there is no such A.

When $z \geq 3$, the least value of xyz is $3 \cdot 4 \cdot 5 = 60$, which is greater than 55. Hence, there is no such a.

Consider the case when $|A| = 4$. Let the element in A be x, y, z, t. WLOG, we can assume $x > y > z > t$. We have $xyzt = 55 - x - y - z - t$.
When $t = 1$, we have $xyz + x + y + z = 54$.
When $z = 2$, we have $2xy + x + y = 52$. Hence, we have $(2x + 1)(2y + 1) = 105$. Since x, y are integers from 1 to 10, we have $x = 7$ and $y = 3$.
When $z \geq 3$, the least value of $xyzt$ is $1 \cdot 3 \cdot 4 \cdot 5 = 60$, which is greater than 55. Hence, there is no such a
When $t \geq 2$, the least value of $xyzt$ is $2 \cdot 3 \cdot 4 \cdot 5 = 120$, which is greater than 55. Hence, there is no such a.

Hence, there are $\boxed{3}$ such A and B. □

Problem 10. During the semester, the students in a math class are divided into groups of four such that every two groups have exactly 2 students in common and no two students are in all the groups together. Find the maximum number of such groups.

Solution. Consider the case when a two-element pair is in more than 3 groups. WLOG, we can have the 1st group to be (A,B,C,D), the 2nd group to be (A,B,E,F), and the 3rd group to be (A,B,G,H), the 4th group to be (A,B,I,J). If the 5th group contains either A or B, it must contain one from CD, one from EF, one from GH, and one from IJ. If the 5th group contains neither A or B, it must contain both CD, EF, GH, and IJ. Hence, any two-element pair should at most be in 3 groups.
Consider the case when a two-element pair is in 3 groups. WLOG, we can have the 1st group to be (A,B,C,D), the 2nd group to be (A,B,E,F), and the 3rd group to be (A,B,G,H). If any other group contains neither A or B, it must contain both CD, EF, and GH, which is impossible. Hence, if a two-element pair is in 3 groups, all groups should contain at least 1 element from it.
A group of 4 elements contains 6 two-element pairs. If each pair can be used only twice, the maximum number of such groups is 7. Assume the number of groups is greater than 7, then we should at least have one pair be in 3 groups. WLOG, we can have the three groups to be (A,B,C,D), (A,B,E,F), and (A,B,G,H). Consider the group (A,B,C,D). Since other groups cannot contain either AB or CD, there are 4 pairs to be used. Since the number of groups is greater than 7, we should at least have other 5 groups. By Pigeonhole Principle, we should have another pair to be in 3 groups. WLOG, we can have the pair to be AC.
In this way, we have pairs AD,and BC remaining, and we should have at least 3 more groups. The 3 more groups cannot contain the same pair, otherwise a pair will be used for four times. Hence, by Pigeonhole Principle, 2 groups should contain the same pair. WLOG, we can have AD to be in 2 groups, then AD will be used three times. Hence, BC cannot be used anymore. In this way, we can't have number of groups greater than 7.
Notice we can have number of groups to be 7.

A	B	C	D	E	F	G	H
√	√	√	√				
√	√			√	√		
√	√					√	√
√		√		√		√	
√		√			√		√
√			√	√			√
√			√		√	√	

Hence, the maximum number of such groups is $\boxed{7}$. □