Team Round

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November 2014

Problem 1. Steven has just learned about polynomials and he is struggling with the following problem: expand $(1-2x)^7$ as $a_0+a_1x+\cdots+a_7x^7$. Help Steven solve this problem by telling him what $a_1+a_2+\cdots+a_7$ is.

Solution. Notice $a_0 = 1$ by binomial theorem. Then, when $x = 1$, we have

$$(1-2)^7 = a_0 + a_1 + \cdots + a_7$$

so $a_0 + a_1 + \cdots + a_7 = -1$. Hence, $a_1 + a_2 + \cdots + a_7 = \boxed{-2}$. □

Problem 2. Each element of the set $\{2, 3, 4, \ldots, 100\}$ is colored. A number has the same color as any divisor of it. What is the maximum number of colors?

Solution. Notice 2 and 3 should have the same color since both of them divide 6. In this way, any prime number less than 50 should have the same color, since we can always find a number less than 100 such that it's the product of 2 and the prime number, so they should have the same color with 2. Hence, any composite number from the set should also have the same color with 2, since they must divide prime numbers less than 11. Thus, numbers can have different colors can only be prime numbers greater than 50, which includes $\{53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$. Hence, the maximum number of colors should be $\boxed{11}$. □

Problem 3. Fuchsia is selecting 24 balls out of 3 boxes. One box contains blue balls, one red balls and one yellow balls. They each have a hundred balls. It is required that she takes at least one ball from each box and that the numbers of balls selected from each box are distinct. In how many ways can she select the 24 balls?

Solution. Let the number of blue balls be x, red balls be y, and yellow balls be z, so $x + y + z = 24$. Let $x' = x - 1$, $y' = y - 1$, and $z' = z - 1$, so $x' + y' + z' = 21$. Since x, y, z must be at least 1, x', y', z' must be at least 0. Then, by stars and bars, we have a total of $\binom{23}{2} = 253$ ways.
When exactly two numbers of balls are the same, the case could be from $(1, 1, 22)$ to $(11, 11, 2)$ except for $(8, 8, 8)$. Hence, there are $10 \cdot 3 = 30$ cases with exactly two numbers are the same and 1 case with three numbers are the same. Thus, there are $253 - 30 - 1 = \boxed{222}$ ways. □

Problem 4. Find the perfect square that can be written in the form $\overline{abcd} - \overline{dcba}$ where a, b, c, d are nonzero digits and $b < c$. $\overline{abcd}$ is the number in base 10 with digits $\overline{abcd}$ written in this order.

Solution. We have

$$(1000a + 100b + 10c + d) - (1000d + 100c + 10b + a) = x^2.$$

Hence, we can have

$$9 \cdot (111(a - d) - 10(c - b)) = x^2.$$

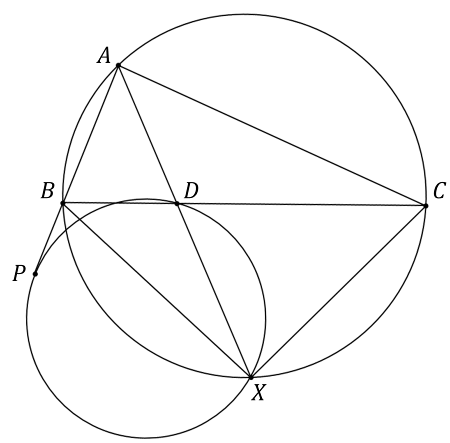
Since 9 is a perfect square, we should have $111(a-d)-10(c-b)$ a perfect square. Since a, b, c, d are nonzero digits and $b < c$, we have $0 < a-d < 9$ and $0 < c-b < 9$.
Since a perfect square always ends with $1, 4, 5, 6, 9$, we have $a-d$ to be either 1, 4, 5, or 6.
When $a-d$ is equal to 1, we have $c-d = 3$ such that $x^2 = 9 \cdot 81$, so $x^2 = 729$.
When $a-d$ is equal to 4, 5, or 6, there is no such c and d.
Hence, the perfect square is $\boxed{729}$. □

Problem 5. Steven has 100 boxes labeled from 1 to 100. Every box contains at most 10 balls. The number of balls in boxes labeled with consecutive numbers differ by 1. The boxes labeled $1, 4, 7, 10, \ldots, 100$ have a total of 301 balls. What is the maximum number of balls Steven can have?

Solution. Since we want maximum number of balls, the balls in the interval 2 boxes should be 10 and 9 each, which results in the balls in boxes with an interval of 3 being 9 and 10, or 9 and 8. The number of balls in boxes labeled $1, 4, 7, \ldots, 100$ will be the greatest when they contain 9 and 10 balls alternatively. In this way, the number of balls will be 323, with 17 boxes with 10 balls and 17 boxes with 9 balls. Notice that we can substitute boxes with 10 balls with 8 balls to reduce the total number of balls by 2. Since there are 17 boxes with 10 balls, we can substitute 11 of them to reduce the number of balls by 22 and end up having 301 balls. Hence,filling the intervals with 10 and 9 each is possible. In this way, we have the maximum number of balls to be $19 \cdot 33 + 301 = \boxed{928}$. □

Problem 6. In acute $\triangle ABC$, $AB = 4$. Let D be the point on BC such that $\angle BAD = \angle CAD$. Let AD intersect the circumcircle of $\triangle ABC$ at X. Let Γ be the circle through D and X that is tangent to AB at P. If $AP = 6$, compute AC.

Solution. Since A, B, C, X are on Γ, we have $\angle ABD = \angle AXC$. Also, since $\angle BAD = \angle XAC$, we have $\triangle ABD$ is similar to $\triangle AXC$. Hence, we have $AC = \frac{AD \cdot AX}{AB}$. Since AP is tangent to AB, we have $AD \cdot AX = AP^2 = 36$, so $AC = \frac{36}{4} = \boxed{9}$. □



Problem 7. Consider a 15×15 square decomposed into unit squares. Consider a coloring of the vertices of the unit squares into two colors, red and blue such that there are 133 red vertices. Out of these 133, two vertices are vertices of the big square and 32 of them are located on the sides of the big square. The sides of the unit squares are colored into three colors. If both endpoints of a side are colored red then the side is colored red. If both endpoints of a side are colored blue then the side is colored blue. Otherwise the side is colored green. If we have 196 green sides, how many blue sides do we have?

Solution. Since there are 256 vertices in total and 133 of them are red, the remaining 123 are blue. Since there are 4 vertices on the vertices of the big square and 2 of them are red, the remaining 2 are blue. Since there are 56 vertices on the sides of the big square and 32 of them are red, the remaining 24 are blue. Hence, the blue vertices inside the square is $123 - 2 - 24 = 97$.
A blue side connects 2 blue vertices and a green side connects 1 blue vertex. A blue vertex on the vertices of the big square connects 2 vertices, on the sides of the big square connects 3 vertices, and inside the big square connects 4 vertices. Let the number of blue sides be x. We have

$$2x + 196 = 2 \cdot 2 + 24 \cdot 3 + 97 \cdot 4.$$

Hence, we have $x = \boxed{134}$. □

Problem 8. Carl has 10 piles of rocks, each pile with a different number of rocks. He notices that he can redistribute the rocks in any pile to the other 9 piles to make the other 9 piles have the same number of rocks. What is the minimum number of rocks in the biggest pile?

Solution. Since each pile has a different number of rocks, if a pile can be redistributed to other 9 piles, it should give the greatest 0, the second greatest 1, etc. In this way, the number of rocks should be at least $1 + 2 + \cdots + 8 = 36$. Hence, the number of rocks should be $36, 37, \ldots, 45$, with 9 piles of 45 rocks after redistribution, so the minimum number of rocks in the biggest pile is $\boxed{45}$. □

Problem 9. Suppose that Tony picks a random integer between 1 and 6 inclusive such that the probability that he picks a number is directly proportional to the the number itself. Danny picks a number between 1 and 7 inclusive using the same rule as Tony. What is the probability that Tony's number is greater than Danny's number?

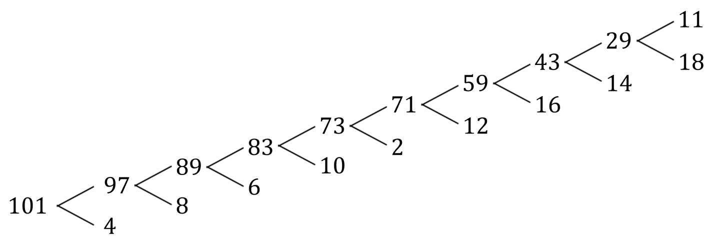
Solution. The probability Tony picks number x is $\frac{x}{21}$. The probability Danny picks number y is $\frac{y}{28}$. When Tony picks number x, Danny should pick any integer between 1 to 7 that is less than x, so the probability that Tony's number is greater than Danny's number is

$$\begin{aligned}\sum_{i=1}^{6}\sum_{j=1}^{i-1}\left(\frac{x}{21} \cdot \frac{y}{28}\right) &= \frac{1 \cdot 0 + 2 \cdot 1 + 3 \cdot 3 + 4 \cdot 6 + 5 \cdot 10 + 6 \cdot 15}{21 \cdot 28} \\ &= \frac{175}{588} \\ &= \boxed{\frac{25}{84}}.\end{aligned}$$

□

Problem 10. Mike wrote on the board the numbers $1, 2, \ldots, n$. At every step, he chooses two of these numbers, deletes them and replaces them with the least prime factor of their sum. He does this until he is left with the number 101 on the board. What is the minimum value of n for which this is possible?

Solution. Since two odd numbers have a sum of even number, they will end up leaving a 2. Hence, to get 101, we must choose an odd number, and other odd numbers will form a 2. Then, we add even number to get prime numbers continuously to get 101.
Notice that when $n = 17$, the maximum it can get is at most $17 + (2 + 4 + 6 + \cdots + 16) = 89$, which is less than 101. Hence, $n \geq 17$.
Consider the case when $n = 18$, we have



Hence, we have $n = \boxed{18}$ to be the minimum. □