Individual Round

Taoyu Pan

November 2015

Problem 1. Find the minimum value of $x^4 + 2x^3 + 3x^2 + 2x + 2$, where x can be any real number.

Solution. Notice that

$$\begin{aligned} x^4 + 2x^3 + 3x^2 + 2x + 2 &= (x^4 + x^3 + x^2) + (x^3 + x^2 + x) + (x^2 + x + 1) + 1 \\ &= (x^2 + x + 1)(x^2 + x + 1) + 1 \\ &= (x^2 + x + 1)^2 + 1. \end{aligned}$$

Hence, $x^4 + 2x^3 + 3x^2 + 2x + 2$ is the minimum if and only if $x^2 + x + 1$ is the minimum. Hence, we have $x = -\frac{1}{2}$, so $x^2 + x + 1 = \frac{3}{4}$ and $x^4 + 2x^3 + 3x^2 + 2x + 2 = \boxed{\dfrac{25}{16}}$. □

Problem 2. A type of digit-lock has 5 digits, each digit chosen from $\{1, 2, 3, 4, 5\}$. How many different passwords are there that have an odd number of 1's?

Solution. Consider the case when there is one 1. There are $\binom{5}{1} = 5$ ways to put the 1. Hence, the number of different passwords is $5 \cdot 4^4 = 1280$.
Consider the case when there are three 1's. There are $\binom{5}{3} = 10$ ways to put the 1's. Hence, the number of different passwords is $10 \cdot 4^2 = 160$.
Consider the case when there are five 1's. There is only one possibility.
Hence, the number of different passwords are $1280 + 160 + 1 = \boxed{1441}$. □

Problem 3. Tony is a really good Ping Pong player, or at least that is what he claims. For him, ping pong balls are very important and he can feel very easily when a ping pong ball is good and when it is not. The Ping Pong club just ordered new balls. They usually order form either PPB company or MIO company. Tony knows that PPB balls have 80% chance to be good balls and MIO balls have 50% chance to be good balls. I know you are thinking why would anyone order MIO balls, but they are way cheaper than PPB balls. When the box full with balls arrives(huge number of balls), Tony tries the first ball in the box and realizes it is a good ball. Given that the Ping Pong club usually orders half of the time from PPB and half of the time from MIO, what is the probability that the second ball is a good ball?

Solution. The probability that the balls are PPB balls is $\frac{80\%}{80\%+50\%} = \frac{8}{13}$, and the probability that the balls are MIO balls is $\frac{50\%}{80\%+50\%} = \frac{5}{13}$. Hence, the probability that the second ball is a good ball is $\frac{8}{13} \times 80\% + \frac{5}{13} \times 50\% = \boxed{\dfrac{89}{130}}$. □

Problem 4. What is the smallest positive integer that is one-ninth of its reverse?

Solution. Notice the integer cannot be one-digit number since the reverse of it is itself.
Consider the case when the integer is two-digit. Let the integer be $\overline{ab}$, we have

$$9(10a + b) = (10b + a).$$

Simplifying the equation, we have

$$89a = b.$$

Since both a and b are one-digit and a is not 0, the case is not possible.
Consider the case when the integer is three-digit. Let the integer be $\overline{abc}$, we have

$$9(100a + 10b + c) = 100c + 10b + a.$$

Simplifying the equation, we have

$$899a + 80b - 91c = 0.$$

Since $9a - c \equiv 0 \mod 10$, $a = 1$ and $c = 9$. Then, $b = -1$, so the case is not possible.
Consider the case when the integer is four-digit. Let the integer be $\overline{abcd}$, we have

$$9(1000a + 100b + 10c + d) = 1000d + 100c + 10b + a.$$

Simplifying the equation, we have

$$8999a + 890b - 10c - 991d = 0.$$

Since $9a - d \equiv 0 \mod 10$, $a = 1$ and $d = 9$. Then, $890b - 10c = -80$, so $b = 0$ and $c = 8$. In this way, $\overline{abcd} = \boxed{1089}$.

□

Problem 5. When Michael wakes up in the morning he is usually late for class so he has to get dressed very quickly. He has to put on a short sleeved shirt, a sweater, pants, two socks and two shoes. People usually put the sweater on after they put the short sleeved shirt on, but Michael has a different style, so he can do it both ways. Given that he puts on a shoe on a foot after he put on a sock on that foot, in how many different orders can Michael get dressed?

Solution. If we ignore the order of shoes and socks, there are $7! = 5040$ different orders. Since the number of orders when Michael puts on a shoe first is the same as the number of orders when he puts on a sock first, and we have 2 pairs of shoes and socks, there are $\frac{5040}{2\times 2} = \boxed{1260}$ different orders. □

Problem 6. The numbers $1, 2, \ldots, 2015$ are written on a blackboard. At each step we choose two numbers and replace them with their nonnegative difference. We stop when we have only one number. How many possibilities are there for this last number?

Solution. There are 1008 odd numbers and 1007 even numbers. If 2 odd numbers are chosen, 1 even number is replaced; if 2 even numbers are chosen, 1 even number is generated; if 1 odd number and 1 even number are chosen, 1 odd number is generated. Hence, we have

number of odd numbers	number of even numbers
-2	+1
-	-1
-	-1

Thus, the oddity of the number of odd numbers stays the same, which implies the last number must be an even number.
Also, for any even number n, we can take the difference of each adjacent pair of numbers with only $n+1$ remains. Hence, there are 1007 number of 1s generated. Then, we can take the difference of 503 pairs of 1s with one 1 remains, and we can take the difference of 1 and $n+1$ to get n. Hence, since the difference is nonnegative, any even number from 0 to 2014 is possible to be the last number. Thus, there are $\boxed{1008}$ possibilities for the last number. □

Problem 7. Let $A = (a_1a_2\ldots a_n)_{34}$ and $B = (b_1b_2\ldots b_n)_{34}$ be two numbers written in base 34. If the sum of the base-34 digits of A is congruent to 15 (mod 77) and the sum of the base-34 digits of B is congruent to 23 (mod 77). Then if $(a_1b_1a_2b_2\ldots a_nb_n)_{34} \equiv x \pmod{77}$ and $0 \le x \le 76$, what is x?(you can write x in base 10)

Solution. We have

$$a_1 + a_2 + \cdots + a_n \equiv 15 \mod 77$$

$$b_1 + b_2 + \cdots + b_n \equiv 23 \mod 77.$$

Expanding $(a_1b_1a_2b_2\ldots a_nb_n)_{34}$, we have

$$(a_1b_1a_2b_2\ldots a_nb_n)_{34} = (a_1 \cdot 34^{2n-1} + a_2 \cdot 34^{2n-3} + \cdots + a_n \cdot 34) + (b_1 \cdot 34^{2n-2} + b_2 \cdot 34^{2n-4} + \cdots + b_n).$$

Notice that $34^2 \equiv 1 (\text{mod } 77)$. Hence, we have

$$(a_1b_1a_2b_2\ldots a_nb_n)_{34} \equiv 34 \cdot (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) \equiv 34 \times 15 + 23 \equiv \boxed{71} \mod 77.$$

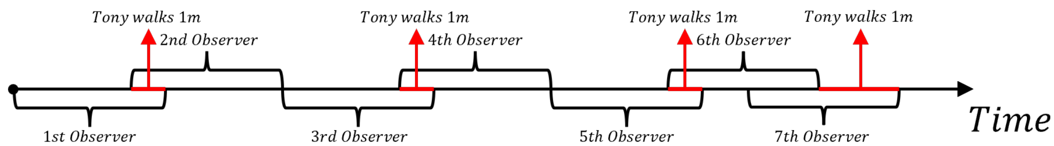
□

Problem 8. What is the sum of the medians of all nonempty subsets of $\{1, 2, \ldots, 9\}$?

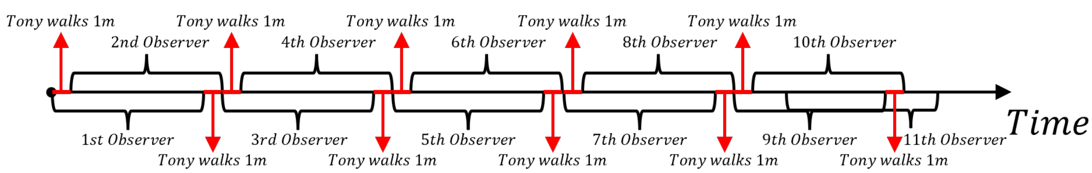
Solution. Since the medians of all nonempty subsets are symmetric about 5, the mean of the medians is 5. Since there are $2^9 - 1 = 511$ nonempty subsets of $\{1, 2, \ldots, 9\}$, the sum of the medians is $5 \times 511 = \boxed{2555}$. □

Problem 9. Tony is moving on a straight line for 6 minutes - classic Tony. Several finitely many observers are watching him because, let's face it, you can't really trust Tony. In fact, they must watch him very closely - so closely that he must never remain unattended for any second. But since the observers are lazy, they only watch Tony uninterruptedly for exactly one minute, and during this minute, Tony covers exactly one meter. What is the sum of the minimal and maximal possible distance Tony can walk during the six minutes?

Solution. In the case when Tony walks the minimal distance, he walks 1 meter during the period when two observers together are watching him. Since each observer watches Tony for 1 minute and two observers should have a time period during which both of them are watching Tony, Tony can walk 1 meter for a period of time slightly less than 2 minutes. In this way, he can walk 3 meters for a period of time slightly less than 6 minutes. However, since 6 meters are not fully covered by the six observers, he should walk 1 more meter when the 7th observer is watching him. Hence, the minimal possible distance Tony can walk is 4 meters.



In the case when Tony walks the maximal distance, he walks 1 meter during the period when observers are watching him individually. Since each observer watches Tony for 1 minute and observers should have a time period during which they are watching Tony individually, Tony can walk 2 meters for a period of time slightly greater than 1 minute. In this way, he can walk 10 meters for a period of time slightly greater than 5 minutes. Also, since the time period during which the 11th observer is watching Tony intersects with the period during which he walks 1 meter as the 10th observer is watching him, Tony cannot walk 1 more meter when the 11th observer is watching him individually. Hence, the maximal possible distance Tony can walk is 10 meters. □



Thus, the sum of the minimal and maximal possible distance Tony can walk during the six minutes is $4 + 10 = \boxed{14}$.

Problem 10. Find the number of nonnegative integer triplets a, b, c that satisfy

$$2^a 3^b + 9 = c^2$$

Solution. We have $2^a 3^b = (c-3)(c+3)$.

Consider the case when $a = 0$ and $b \neq 0$. We have $3^b = (c-3)(c+3)$. Hence, we should have two powers of 3 that have a difference of 6. Notice that differences of adjacent powers of 3 are greater than 6 after 9, so we can only have $c = 6$ such that $3^b = 3 \times 9$, so $b = 3$. Thus, we have a triplet $(0, 3, 6)$ in this case.

Consider the case when $a \neq 0$ and $b = 0$. We have $2^a = (c-3)(c+3)$. Hence, we should have two powers of 2 that have a difference of 6. Notice that differences of adjacent powers of 2 are greater than 6 after 8, so we can only have $c = 5$ such that $2^a = 2 \times 8$, so $a = 4$. Thus, we have a triplet $(4, 0, 5)$ in this case.

Consider the case when $a \neq 0$ and $b \neq 0$. We have $2^a 3^b = (c-3)(c+3)$. Hence, we have $(c-3)(c+3) \equiv 0 \mod 6$, so $c \equiv 3 \mod 6$. Thus, we can have $c = 6k+3$, and $2^a 3^b = (6k)(6k+6) = 36k(k+1)$. Hence, we have $2^{a-2}3^{b-2} = k(k+1)$. Since k and $k+1$ are coprime, it is whether

$$\begin{cases} 2^{a-2} = k \\ 3^{b-2} = k+1 \end{cases} \quad \text{or} \quad \begin{cases} 2^{a-2} = k+1 \\ 3^{b-2} = k \end{cases}$$

Consider the case when $2^{a-2} = k$ and $3^{b-2} = k+1$. We have $2^{a-2} + 1 = 3^{b-2}$.
When $a < 4$, we have $a = 3$ and $b = 3$, so $k = 2$ and $c = 15$. Thus, we have a triplet $(3, 3, 15)$ in this case.
When $a \geq 4$, we have $3^{b-2} \equiv 1 \mod 4$, so $b = 2n$. Hence, we have

$$3^{b-2} = 3^{2(n-1)} - 1 = (3^{n-1} - 1)(3^{n-1} + 1).$$

Thus, we have $2^{a-2} = (3^{n-1} - 1)(3^{n-1} + 1)$, so we should have two powers of 2 that have a difference of 2. Notice that differences of adjacent powers of 2 are greater than 2 after 4, so we can only have $n = 2$ such that $2^{a-2} = 2 \times 4$, so $a = 5$ and $b = 4$. Thus, we have $k = 8$ and $c = 51$, so we have a triplet $(5, 4, 51)$ in this case.
Consider the case when $2^{a-2} = k+1$ and $3^{b-2} = k$. We have $2^{a-2} = 3^{b-2} + 1$.
When $b < 3$, we have $b = 2$ and $a = 3$, so $k = 1$ and $c = 9$. Thus, we have a triplet $(3, 2, 9)$ in this case.
When $b \geq 3$, we have $2^{a-2} \equiv 1 \mod 3$, so $a = 2m$. Hence, we have

$$2^{a-2} = 2^{2(m-1)} - 1 = (2^{m-1} - 1)(2^{m-1} + 1).$$

Thus, we have $3^{b-2} = (2^{m-1} - 1)(2^{m-1} + 1)$, so we should have two powers of 3 that have a difference of 2. Notice that differences of adjacent powers of 3 are greater than 2 after 3, so we can only have $m = 2$ such that $3^{b-2} = 1 \times 3$, so $a = 4$ and $b = 3$. Thus, we have $k = 3$ and $c = 21$, so we have a triplet $(4, 3, 21)$ in this case.

Hence, triplets are $(0, 3, 6), (4, 0, 5), (3, 3, 15), (5, 4, 51), (3, 2, 9), (4, 3, 21)$. Thus, the number of triplets is $\boxed{6}$. $\square$