Team Round

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Problem 1. Let $U = \{-2, 0, 1\}$ and $N = \{1, 2, 3, 4, 5\}$. Let f be a function that maps U to N. For any $x \in U$, $x + f(x) + xf(x)$ is an odd number. How many f satisfy the above statement?

Solution. Notice that when $x + f(x) + xf(x)$ is an odd number, at least one of x and $f(x)$ should be odd. Hence, when $x = -2$ and $x = 0$, there are 3 possible values of $f(x)$. When $x = 1$, there are 5 possible values of $f(x)$. Hence, the number of possible f is $3 \times 3 \times 5 = \boxed{45}$. □

Problem 2. Around a circle are written all of the positive integers from 1 to n, $n \geq 2$ in such a way that any two adjacent integers have at least one digit in common in their decimal expressions. Find the smallest n for which this is possible.

Solution. Notice $n \geq 9$ since no one-digit number has a common digit with another. Hence, 1 to n must contain 9. The least two numbers that have one digit in common with 9 are 19 and 29, so we have $n \geq 29$. When $n = 29$, we can construct the circle by grouping up every one-digit number $\overline{A}$ in the middle of $\overline{1A}$ and $\overline{2A}$. Then, we can have the circle to be $\{\cdots, \overline{1A}, \overline{A}, \overline{2A}, \overline{2B}, \overline{B}, \overline{1B}, \cdots\}$. Since we also have 10 and 20, there are 10 such groups, which makes the beginning and end possible to be adjacent. Hence, we have the smallest $n = \boxed{29}$. □

Problem 3. Michael loses things, especially his room key. If in a day of the week he has n classes he loses his key with probability $\frac{n}{5}$. After he loses his key during the day he replaces it before he goes to sleep so the next day he will have a key. During the weekend(Saturday and Sunday), Michael studies all day and does not leave his room, therefore he does not lose his key. Given that on Monday he has 1 class, on Tuesday and Thursday he has 2 classes and that on Wednesday and Friday he has 3 classes, what is the probability that loses his key at least once during a week?

Solution. If Michael does not lose his key during a week, the probability is

$$(1 - \frac{1}{5}) \cdot (1 - \frac{2}{5})^2 \cdot (1 - \frac{3}{5})^2 = \frac{144}{3125}.$$

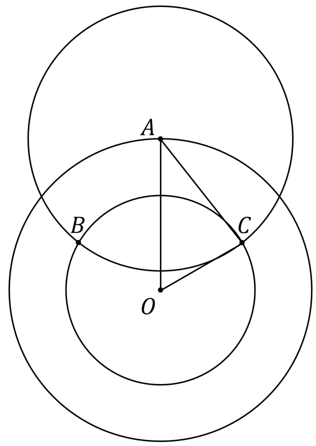
Hence, the probability that Michael loses his key at least once during a week is

$$1 - \frac{144}{3125} = \boxed{\frac{2981}{3125}}.$$

□

Problem 4. Given two concentric circles one with radius 8 and the other 5. What is the probability that the distance between two randomly chosen points on the circles, one from each circle, is greater than 7?

Solution. We have $AO = 8$, $OC = 5$, and $AC = 7$.



By Heron's Formula, we have the area of $\triangle AOC = \sqrt{10 \times 2 \times 5 \times 3} = 10\sqrt{3}$. Hence, we have

$$\frac{1}{2} \times 8 \times 5 \times \sin \angle AOC = 10\sqrt{3}.$$

Hence, $\sin \angle AOC = \frac{\sqrt{3}}{2}$, so $\angle AOC = \frac{1}{3}\pi$. Thus, the probability is $1 - \frac{2 \times \frac{1}{3}\pi}{2\pi} = \boxed{\frac{2}{3}}$. □

Problem 5. We say that a positive integer n is lucky if n^2 can be written as the sum of n consecutive positive integers. Find the number of lucky numbers strictly less that 2015.

Solution. Notice that if $n = 2k + 1$, we can have

$$\sum_{i=k+1}^{3k+1} i = \frac{(4k+2)(2k+1)}{2} = (2k+1)^2 = n^2.$$

Hence, every odd integer is lucky.
If $n = 2k$, then $n^2 = 4k^2$. If $4k^2$ can be written as the sum of $2k$ consecutive positive integers, we should have the kth and $(k+1)$th number to have a sum of $4k$. Since two consecutive positive integers will have an odd sum, the case is not possible.
Hence, a number is lucky if and only if it's odd, then there are $\boxed{1007}$ lucky numbers strictly less than 2015. □

Problem 6. Let $A = \{3^x + 3^y + 3^z | x, y, z \geq 0, x, y, z \in \mathbb{Z}, x < y < z\}$. Arrange the set A in increasing order. Then what is the 50th number? (Express the answer in the form $3^x + 3^y + 3^z$).

Solution. Set A is the set of all integers with exactly three 1s and no 2 when written in base 3. Hence, we only need to count the 50th smallest number with this property.
When there is no 0 when written in base 3, there is only 1 such number.
When there is one 0 when written in base 3, by Bars and Stars, we have $\binom{4}{1} = 4$ numbers in total, and $\binom{3}{0} = 1$ number starting with 0. Hence, there are $4 - 1 = 3$ such numbers.
Similarly, when there are two 0s when written in base 3, there are $\binom{5}{2} - \binom{4}{1} = 10 - 4 = 6$ such numbers.

Similarly, when there are three 0s when written in base 3, there are $\binom{6}{3} - \binom{5}{2} = 20 - 10 = 10$ such numbers.
Similarly, when there are four 0s when written in base 3, there are $\binom{7}{4} - \binom{6}{3} = 15$ such numbers.
Similarly, when there are four 0s when written in base 3, there are $\binom{8}{5} - \binom{7}{4} = 21$ such numbers.
Since $1 + 3 + 6 + 10 + 15 + 21 = 56$, the 50th smallest number will be the 7th greatest when there are five 0s when written in base 3. Hence, we have the number to be $\boxed{3^4 + 3^5 + 3^7}$. □

Problem 7. Justin and Oscar found 2015 sticks on the table. I know what you are thinking, that is very curious. They decided to play a game with them. The game is, each player in turn must remove from the table some sticks, provided that the player removes at least one stick and at most half of the sticks on the table. The player who leaves just one stick on the table loses the game. Justin goes first and he realizes he has a winning strategy. How many sticks does he have to take off to guarantee that he will win?

Solution. Notice that the player who is left with 2 sticks will lose the game, since the player must remove exactly 1 stick and leave 1 stick on the table. Hence, the player who is left with $2 \times 2 + 1 = 5$ sticks will also lose the game, since the number of sticks left will be from 3 to 4, so the other player can always leave 2 sticks. In this way, the player who is left with $5 \times 2 + 1 = 11$ sticks will also lose the game, since the number of sticks left will be from 6 to 10, so the other player can always leave 5 sticks.
Thus, the number of sticks left should be a sequence such that $a_n = 2a_{n-1} + 1$. Hence, $a_n + 1 = 2(a_{n-1} + 1)$, so $a_n = (a_1 + 1) \cdot 2^{n-1} - 1$. Since $a_1 = 2$, we have $a_n = 3 \cdot 2^{n-1} - 1$. When $n = 9$, we have $a_9 = 1535$, so Justin has to take off $2015 - 1535 = \boxed{480}$ sticks to guarantee that he will win. □

Problem 8. Let (x, y, z) with $x \geq y \geq z \geq 0$ be integers such that $\frac{x^3+y^3+z^3}{3} = xyz + 21$. Find x.

Solution. Simplifying the equation, we have $x^3 + y^3 + z^3 - 3xyz = 63$. Notice that

$$\begin{aligned} & x^3 + y^3 + z^3 - 3xyz \\ =& (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz) \\ =& (x + y + z) \cdot \frac{(x - y)^2 + (x - z)^2 + (y - z)^2}{2}. \end{aligned}$$

Hence, we have

$$(x + y + z)((x - y)^2 + (x - z)^2 + (y - z)^2) = 126.$$

Since $126 = 2 \times 3^2 \times 7$, one of $x + y + z$ and $(x - y)^2 + (x - z)^2 + (y - z)^2$ should be even and the other should be odd. When $x + y + z$ is even, it is whether three of them are even or one of them is even and the other two are odd. When three of them are even, $(x - y)^2 + (x - z)^2 + (y - z)^2$ is also even, which is not possible. When one of them is even and the other two are odd, $(x - y)^2 + (x - z)^2 + (y - z)^2$ is also even, which is not possible.
Hence, the only possible case is when $x + y + z$ is odd and $(x - y)^2 + (x - z)^2 + (y - z)^2$ is even. When $x + y + z$ is odd, it is whether three of them are odd or one of them is odd and the other two are even. When three of them are odd, $(x - y)$, $(x - z)$, and $(y - z)$ are all even, so $(x - y)^2 + (x - z)^2 + (y - z)^2$ is a multiple of 4, which is not possible. Hence, the only possible case is when one of x, y, z is odd and the other two are even.
The odd factors of 126 are $1, 3, 7, 9, 21, 63$.
When $x + y + z = 1$, the only possible case is $(1, 0, 0)$, which is not possible.
When $x + y + z = 3$, the possible cases are $(3, 0, 0), (2, 1, 0)$, which are not possible.
When $x + y + z = 7$, the possible cases are $(7, 0, 0), (5, 2, 0), (4, 3, 0), (3, 2, 2), (6, 1, 0), (4, 2, 1)$, which are not possible.
When $x+y+z = 9$, the possible cases are $(9, 0, 0), (7, 2, 0), (5, 4, 0), (5, 2, 2), (6, 3, 0), (4, 3, 2), (8, 1, 0), (6, 2, 1)$, $(4, 4, 1)$, which are not possible.
When $x+y+z = 21$, we have $(x-y)^2+(x-z)^2+(y-z)^2 = 6$, so the only possible case is when $x-z = 2$,

$x - y = 1$, and $y - z = 1$. Hence, we have a possible case of $(8, 7, 6)$.
When $x + y + z = 63$, we have $(x-y)^2 + (x-z)^2 + (y-z)^2 = 2$, so the only possible case is when $x - z = 1$, $x - y = 1$, and $y - z = 0$. Hence, we have $x + (x-1) + (x-1) = 63$, which is not possible.
Thus, we have $x = \boxed{8}$. □

Problem 9. Let $p < q < r < s$ be prime numbers such that

$$1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} - \frac{1}{s} = \frac{1}{pqrs}.$$

Find $p + q + r + s$.

Solution. If $p > 2$, we have $\frac{1}{pqrs} \le \frac{1}{3\times5\times7\times11} = \frac{1}{1155}$, and $1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} - \frac{1}{s} \ge 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{11} = \frac{269}{1155}$. In this case, we always have $1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} - \frac{1}{s} > \frac{1}{pqrs}$, which is not possible. Thus, we must have $p = 2$, then we have

$$\frac{1}{2} - \frac{1}{q} - \frac{1}{r} - \frac{1}{s} = \frac{1}{2qrs}.$$

If $q > 3$, we have $\frac{1}{2qrs} \le \frac{1}{2\times5\times7\times11} = \frac{1}{770}$, and $\frac{1}{2} - \frac{1}{q} - \frac{1}{r} - \frac{1}{s} \ge \frac{1}{2} - \frac{1}{5} - \frac{1}{7} - \frac{1}{11} = \frac{51}{770}$. In this case, we always have $\frac{1}{2} - \frac{1}{q} - \frac{1}{r} - \frac{1}{s} > \frac{1}{2qrs}$, which is not possible. Thus, we must have $q = 3$, then we have

$$\frac{1}{6} - \frac{1}{r} - \frac{1}{s} = \frac{1}{6rs}.$$

Simplifying the equation, we have $rs - 6s - 6r = 1$, so $rs - 6s - 6r + 36 = 37$, which indicates

$$(r-6)(s-6) = 37.$$

Since 37 is a prime number, we have $r - 6 = 1$ and $s - 6 = 37$, so $r = 7$ and $s = 43$.
Hence, we have $p + q + r + s = 2 + 3 + 7 + 43 = \boxed{55}$. □

Problem 10. In "island-land", there are 10 islands. Alex falls out of a plane onto one of the islands, with equal probability of landing on any island. That night, the Chocolate King visits Alex in his sleep and tells him that there is a mountain of chocolate on one of the islands, with equal probability of being on each island. However, Alex has become very fat from eating chocolate his whole life, so he can't swim to any of the other islands. Luckily, there is a teleporter on each island. Each teleporter will teleport Alex to exactly one other teleporter (possibly itself) and each teleporter gets teleported to by exactly one teleporter. The configuration of the teleporters is chosen uniformly at random from all possible configurations of teleporters satisfying these criteria. What is the probability that Alex can get his chocolate?

Solution. Since each teleporter will teleport Alex to exactly one other teleporter and each teleporter gets teleported to by exactly one teleporter, the configuration of the teleporters will consist of several rings. Alex can get his chocolate if and only if he falls onto one of the islands which is in the same ring as the island with chocolate mountain.
The probability that the chocolate mountain has a ring of size n is

$$\left(\prod_{i=1}^{n-1} \frac{10-i}{11-i}\right) \cdot \frac{1}{11-n} = \frac{1}{10}.$$

For example, the probability that the chocolate mountain has a ring of size 3 is $\frac{9}{10} \times \frac{8}{9} \times \frac{1}{8} = \frac{1}{10}$, since the first island can teleport Alex to any of the island besides itself, and the second island can teleport Alex to any of the island besides the first island and itself, and the third island must teleport Alex to the first island.

Hence, there is a $\frac{1}{10}$ probability for any of the chocolate mountain that has a ring size of n, and the probability Alex falls onto this ring is $\frac{n}{10}$. Hence, the probability that Alex can get his chocolate is

$$\frac{1}{10} \cdot \sum_{i=1}^{10} \frac{i}{10} = \boxed{\frac{11}{20}}.$$

□