Duke Math Meet 2016

Individual Solutions

1. His total after 4 tests was $3(60)+50=230$. In order to have average of 60, his final score needs to be 300. So the fifth test must be $300-230=\boxed{70}$.

2. $20a+16b=2016-ab \implies ab+20a+16b=2016$. Note that the left hand side is $(a+16)(b+20)-20(16)$ (common factoring technique). So we have $(a+16)(b+20)=2016+20(16)=2336$. 2336 factors as $2^5\times 73$. So it has $6\times 2=12$ positive divisors. Note that every pair of divisors gives us a solution (subtract 16 from first and 20 from second). However the question asks for integers, which includes negative divisors. So we have a total $=2\times 12=\boxed{24}$ solutions

3. $f(2t)=f(t)+t$. So there are t values between $f(t)$ and $f(2t)$ $(f(2t)-f(t)=t$. But there are t values between t and $2t$. Hence we see that $f(t+x)=f(t)+x$. So $f(2016)=f(1)+2015=\boxed{4031}$. Making this argument rigorous can be done by induction.

4. Let A,B,C,D be the centers of the circles with radius $7,7,18,r$ respectively. So $AC=BC=25$. Let E be the midpoint of AB. So $AE=7$ and thus by Pythagorean theorem and symmetry, $CE=24$. Drawing a picture, we see that D must lie in the small hole between the circles. So we have $AD=7+r$, $CD=18+r$.

   $DE+CD=CE=24$ and $DE^2+AE^2=AD^2$. Hence $r^2+14r=DE^2=(24-CD)^2=(24-(18+r))^2=(6-r)^2=36-12r+r^2$. Solving this, we see that $r=\frac{18}{13}$. So the answer is $18+13=\boxed{31}$.

5. If we draw the perpendicular bisectors of the three sides of the triangle, then the area of the sector closest to 45^o angle is what we want. Note that intersection of perpendicular bisectors is the center of the circle. To find the area relative to whole circle, we just need to find the angle. The quadrilateral formed has three other angles of total $90^o+90^o+45^o=225^o$. Thus the last angle is $360^o-225^o=135^o$. So the area relative to whole circle is equal to $\frac{135}{360}=\boxed{\frac{3}{8}}$.

6. Any common divisor of a,b divides p. So the gcd of a,b is either 1 or p. If the gcd is p, then $a=pk$ and $b=p(k-1)$ but $ab=p^2k(k-1)$ is only perfect square if $k,k-1$ are perfect squares. But that's not possible.

   Hence the gcd is 1 and if ab is a perfect square, then a,b are individually perfect squares. So we have $a=x^2, b=y^2$ and $x^2-y^2=(x-y)(x+y)=p$. But p only has one pair of divisors $(1,p)$ so $x-y=1$ and $x+y=p$. Solving these for x, we get $x=\frac{p+1}{2}$. In order for $a=x^2<100$, then $x\le 9$. However we see that if $x=9$, then $a=81$ and $b=64$ and we can check that these work. Hence $\boxed{81}$ is the largest solution.

7. 18 lines so $\binom{18}{3}$ ways to form a possible triangles. We must take out all triples from 2 or 3 parallel lines: $2(\binom{6}{2}12+\binom{6}{3})$. Hence, $816-2(180+20)=\boxed{416}$.

8. $(OD^2+OE^2+OF^2)(a^2+b^2+c^2) \geq (a \times OD + b \times OE + c \times OF)^2 = (2S_{ABC})^2$. But $a^2+b^2+c^2 = 5^2+4^2+3^2 = 50$ and $2S_{ABC} = 3(4) = 12$ (right triangle). So we have $\frac{144}{50} = \boxed{\frac{72}{25}}$.

9. Note that $x = 0$ is not a root. Divide by x^2. We get $x^2 - 3x + 3x^{-1} + x^{-2}$. We can group this as $(x^2+x^{-2}) - 3(x-x^{-1}) = (x-x^{-1})^2 + 2 - 3(x-x^{-1})$. If we let $y = x - x^{-1}$, we have $y^2-3y+2 = (y-2)(y-1) = 0$. So we have $x - x^{-1} = 2 \implies x^2-2x-1 = (x-1)^2-2 = 0$. So we have root $1 \pm \sqrt{2}$. $x - x^{-1} = 1 \implies x^2 - x - 1 = (x-\frac{1}{2})^2 - \frac{5}{4} = 0$. Which gives us roots $\frac{1\pm\sqrt{5}}{2}$. Using estimates of square roots, it is easy to see that $\boxed{1+\sqrt{2}}$ is the largest real root.

10. We first find the probability that (x, y) appears. Given the position x, we need y to appear opposite it. This has probability $\frac{1}{7}$. We then calculate the expected value of $|x-y|$. $|x-y| = n$ when $x = i, y = i+n$ or $x = i+n, y = i$ where $1 \leq x, y \leq 8$. Smallest value of i is 1 and $i + n \leq 8$ so $i \leq 8 - n$. Hence $|x-y| = n$ a total of $2(8-n)$ times. $\sum_{i=1}^{7} 2n(8-n) = 84$. So we have total expected value of $\frac{84}{7} = \boxed{12}$.