

Duke Math Meet 2017

Individual Round

October 31, 2017

1. How many subsets of $\{D, U, K, E\}$ have an odd number of elements?
2. Find the coefficient of x^{12} in $(1 + x^2 + x^4 + \dots + x^{28})(1 + x + x^2 + \dots + x^{14})^2$.
3. How many 4-digit numbers have their digits in non-decreasing order from left to right?
4. A dodecahedron (a polyhedron with 12 faces, each a regular pentagon) is projected orthogonally onto a plane parallel to one of its faces to form a polygon. Find the measure (in degrees) of the largest interior angle of this polygon.
5. Justin is back with a 6×6 grid made of 36 colorless squares. Dr. Kraines wants him to color some squares such that
 - Each row and column of the grid must have at least one colored square
 - For each colored square, there must be another colored square on the same row or columnWhat is the minimum number of squares that Justin will have to color?
6. Inside a circle C , we have three equal circles C_1, C_2, C_3 , which are pairwise externally tangent to each other and all internally tangent to C . What is the ratio of the area of C_1 to the area of C ?
7. There are 3 different paths between the Duke Chapel and the Physics building. 6 students are heading towards the Physics building for a class, so they split into 3 pairs and each pair takes a separate path from the Chapel. After class, they again split into 3 pairs and take separate paths back. Find the number of possible scenarios where each student's companion on the way there is different from their companion on the way back.
8. Let a_n be a sequence that satisfies the recurrence relation

$$a_n a_{n+2} = \frac{\cos(3a_{n+1})}{\cos(a_{n+1})[2\cos(2a_{n+1}) - 1]} a_{n+1}$$

with $a_1 = 2$ and $a_2 = 3$. Find the value of 2018 a_{2017} .

9. Let $f(x)$ be a polynomial with minimum degree, integer coefficients, and leading coefficient of 1 that satisfies $f(\sqrt{7} + \sqrt{13}) = 0$. What is the value of $f(10)$?
10. 1024 Duke students, indexed 1 to 1024, are having a chat. For each $1 \leq i \leq 1023$, student i claims that student $2^{\lfloor \log_2 i \rfloor + 1}$ has a girlfriend. ($\lfloor x \rfloor$ is the greatest integer less than or equal to x .) Given that exactly 201 people are lying, find the index of the 61st liar (ordered by index from smallest to largest).