Power Round

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1 SG Sequences

Definition 1. Let $a_1, a_2, \cdots, a_n$ be a sequence of positive integers. We say that the sequence is **semi-geometric** if for all $i = 1, 2, \cdots, n-1$, there exists and integer $k_i > 1$ such that $k_i a_i = a_{i+1}$. For instance, 2,6,12,24 is semi-geometric, whereas 2,3,6,12 is not since 2 does not divide 3. We call all such sequences semi-geometric sequences, or **SG sequences**.

Now that we've defined what SG sequences are, we will do some exercises that will help you better understand the concept.

Problem 1 (4 points, 2 points each). Answering the following questions.

(a) Compute all SG sequences of length 5 such that $a_5 = 84$.

Solution. Since we want SG sequences of length 5, there should be 4 values of k_i, and we must have $a_1 \times \prod_{i=1}^{4} k_i = 84$. Since $84 = 2^2 \times 3 \times 7$, we must have a_1 to be 1 and k_i to be one permutation of $\{2, 2, 3, 7\}$. In this way, we can compute all SG sequences: □

$\{1, 2, 4, 12, 84\}$	$\{1, 3, 6, 12, 84\}$	$\{1, 7, 14, 28, 84\}$
$\{1, 2, 4, 28, 84\}$	$\{1, 3, 6, 42, 84\}$	$\{1, 7, 14, 42, 84\}$
$\{1, 2, 6, 12, 84\}$	$\{1, 3, 21, 42, 84\}$	$\{1, 7, 21, 42, 84\}$
$\{1, 2, 6, 42, 84\}$		
$\{1, 2, 14, 28, 84\}$		
$\{1, 2, 14, 42, 84\}$		

(b) Compute all SG sequences of length 3 such that $a_3 = 36$.

Solution. Since we want SG sequences of length 3, there should be 2 values of k_i, and we must have $a_1 \times \prod_{i=1}^{2} = 36$. Since $36 = 2^2 \times 3^2$, we can compute all SG sequences: □

$\{1, 2, 36\}$	$\{2, 4, 36\}$	$\{3, 6, 36\}$	$\{6, 12, 36\}$	$\{4, 12, 36\}$	$\{9, 18, 36\}$
$\{1, 4, 36\}$	$\{2, 6, 36\}$	$\{3, 12, 36\}$	$\{6, 18, 36\}$		
$\{1, 3, 36\}$	$\{2, 18, 36\}$	$\{3, 9, 36\}$			
$\{1, 9, 36\}$	$\{2, 12, 36\}$	$\{3, 18, 36\}$			
$\{1, 6, 36\}$					
$\{1, 12, 36\}$					
$\{1, 18, 36\}$					

Problem 2 (4 points, 2 points each). Let $a_1, \cdots, a_n$ be an SG sequence such that $a_n = 50400$.

(a) What is the largest possible value for n? (In other words, what is the length of the longest possible sequence with last term equal to 50400?)

Solution. Since $50400 = 2^5 \times 3^2 \times 5^2 \times 7$, we can have $a_1 = 1$ and k_i be the prime factors of 50400. Hence, the largest possible value for n would be $(5 + 2 + 2 + 1) + 1 = 11$. □

(b) Let k be the answer you receive from part a. What is the number of distinct SG sequences of length k such that $a_k = 50400$?

Solution. Since $a_1 = 1$ and we have k_i to be one permutation of 5 twos, 2 threes, 2 fives, and 1 seven, the number of distinct SG sequences of length 11 is $\frac{10!}{5! \cdot 2! \cdot 2!} = 7560$. □

Problem 3 (3 points). Find the number of distinct SG sequences whose last term equals 420. (Note: the sequence (420) is also an SG sequence of length 1)

Solution. Let $A_{n,k}$ denote the number of sequences with length n whose first term equals k and last term equals 420. Let S_n denote the total number of sequences with length n whose last term equals 420. Hence, we have

$$S_n = \sum_{i|420} A_{n,i}.$$

Also, since we can add i at the beginning of the sequence starting with j if $i|j$ and $i < j$, we have

$$A_{n+1,i} = \sum_{i|j;i<j} A_{n,j}.$$

Since $420 = 2^2 \times 3 \times 5 \times 7$, we have $1 \leq n \leq 6$ and all divisors of 420 are

$$\{1, 2, 3, 4, 5, 6, 7, 10, 12, 14, 15, 20, 21, 28, 30, 35, 42, 60, 70, 84, 105, 140, 210, 420\}.$$

Hence, when $n = 1$, we have

$$\begin{array}{llll} A_{1,1} = 0 & A_{1,2} = 0 & A_{1,3} = 0 & A_{1,4} = 0 \\ A_{1,5} = 0 & A_{1,6} = 0 & A_{1,7} = 0 & A_{1,10} = 0 \\ A_{1,12} = 0 & A_{1,14} = 0 & A_{1,15} = 0 & A_{1,20} = 0 \\ A_{1,21} = 0 & A_{1,28} = 0 & A_{1,30} = 0 & A_{1,35} = 0 \\ A_{1,42} = 0 & A_{1,60} = 0 & A_{1,70} = 0 & A_{1,84} = 0 \\ A_{1,105} = 0 & A_{1,140} = 0 & A_{1,210} = 0 & A_{1,420} = 1 \end{array}$$

Hence, $S_1 = 1$.
When $n = 2$, we have

$$\begin{array}{llll} A_{2,1} = 1 & A_{2,2} = 1 & A_{2,3} = 1 & A_{2,4} = 1 \\ A_{2,5} = 1 & A_{2,6} = 1 & A_{2,7} = 1 & A_{2,10} = 1 \\ A_{2,12} = 1 & A_{2,14} = 1 & A_{2,15} = 1 & A_{2,20} = 1 \\ A_{2,21} = 1 & A_{2,28} = 1 & A_{2,30} = 1 & A_{2,35} = 1 \\ A_{2,42} = 1 & A_{2,60} = 1 & A_{2,70} = 1 & A_{2,84} = 1 \\ A_{2,105} = 1 & A_{2,140} = 1 & A_{2,210} = 1 & \end{array}$$

Hence, $S_2 = 23$.
When $n = 3$, we have

$$\begin{array}{llll} A_{3,1} = 22 & A_{3,2} = 14 & A_{3,3} = 10 & A_{3,4} = 6 \\ A_{3,5} = 10 & A_{3,6} = 6 & A_{3,7} = 10 & A_{3,10} = 6 \\ A_{3,12} = 2 & A_{3,14} = 6 & A_{3,15} = 4 & A_{3,20} = 2 \\ A_{3,21} = 4 & A_{3,28} = 2 & A_{3,30} = 2 & A_{3,35} = 4 \\ A_{3,42} = 2 & A_{3,70} = 2 & A_{3,105} = 1 & \end{array}$$

Hence, $S_3 = 115$.

When $n = 4$, we have

$$\begin{array}{llll} A_{4,1} = 93 & A_{4,2} = 36 & A_{4,3} = 21 & A_{4,4} = 6 \\ A_{4,5} = 21 & A_{4,6} = 6 & A_{4,7} = 21 & A_{4,10} = 6 \\ A_{4,14} = 6 & A_{4,15} = 3 & A_{4,21} = 3 & A_{4,35} = 3 \end{array}$$

Hence, $S_4 = 225$.
When $n = 5$, we have

$$A_{5,1} = 132 \quad A_{5,2} = 24 \quad A_{5,3} = 12 \quad A_{5,5} = 12 \quad A_{5,7} = 12$$

Hence, $S_5 = 192$.
When $n = 6$, we have

$$A_{6,1} = 60$$

Hence, $S_6 = 60$.
Thus, the total number of distinct SG sequences is $1 + 23 + 115 + 225 + 192 + 60 = 616$. □

2 SG Numbers

In this section, we will explore some interesting properties of such SG sequences, namely the *SG numbers*.

Definition 2. For each positive integer n, we define the **SG number** of n to be the length of the longest SG sequence $a_1, \cdots, a_k$ such that $a_1 + \cdots + a_k = n$. In this particular case, we denote $\sigma(n) = k$. (For instance, $\sigma(9) = 3$ since the SG sequence 1,2,6 adds up to 9, whereas there cannot be an SG sequence with length 4 that add up to 9. This is because for an SG sequence to add up to 9, its first term must divide 9, hence it must be either 1, 3, or 9. If it's 9, it has length of 1. If it's 3, the only possible SG sequence is 3,6, which has length of 2. If it's 1, then the remaining sequence can be either 8 or 2,6, which implies that 1,2,6 is the longest SG sequence that adds up to 9, hence $\sigma(9) = 3$.)

Here are some exercises that will help us familiarize ourselves with SG numbers.

Problem 4 (12 points, 3 points each). Find, with proof, the following values:

(a) $\sigma(19)$.

Solution. Note that $19 = 1 + 2 + 4 + 12$, so $\sigma(19) \geq 4$. The least number n such that $\sigma(n) = 5$ will be $1 + 2 + 4 + 8 + 16 = 31$, so $\sigma(19) = 4$. □

(b) $\sigma(25)$.

Solution. The first term of the SG sequence must divide 25, so it must be either 1, 5, or 25.
If it's 25, it has length of 1.
If it's 5, the SG sequence can only be $4, 20$.
If it's 1, then the remaining sequence can be $\{24\}$, $\{2, 22\}$, $\{3, 21\}$, $\{4, 20\}$, $\{6, 18\}$, or $\{8, 16\}$, which implies the longest SG sequence has a length of 3, hence $\sigma(25) = 3$. □

(c) $\sigma(95)$.

Solution. Note that $95 = 1+2+4+8+16+64$, so $\sigma(95) \geq 6$. The least number n such that $\sigma(n) = 7$ will be $1+2+4+8+16+32+64 = 127$, so $\sigma(95) = 6$. □

(d) $\sigma(100)$.

Solution. The first term of the SG sequence must divide 100, so it must be either 1, 2, 4, 5, 10, 20, 25, 50, or 100.
If it's 100, it has a length of 1.
If it's 50, it cannot form an SG sequence.
If it's 25, the only possible SG sequence is $\{25, 75\}$, which has a length of 2.
If it's 20, the only possible SG sequence is $\{20, 80\}$, which has a length of 2.
If it's 10, then the remaining sequence can be either $\{90\}$ or $\{30, 60\}$, which implies the longest SG sequence has a length of 3.
If it's 5, the only possible SG sequence is $\{5, 95\}$, which has a length of 2.
If it's 4, then the longest SG sequence has a length of $\sigma(25) = 3$.
If it's 2, then the remaining sequence can either be $\{98\}$, $\{14, 84\}$, or $\{14, 28, 56\}$, which implies the longest sequence has a length of 4.
If it's 1, then the next term must divide 99 and greater than 1, so it must be 3, 9, 11, 33, or 99. If it's 99, it has a length of 2. If it's 33, the only possible SG sequence is $\{1, 33, 66\}$, which has a length of 3. If it's 11, the sequence can either be $\{1, 11, 88\}$ or $\{1, 11, 22, 66\}$, which implies the longest SG sequence has a length of 4. If it's 9, the sequence can either be $\{1, 9, 90\}$ or $\{1, 9, 18, 72\}$, which implies the longest SG sequence has a length of 4. If it's 3, the sequence can either be $\{1, 3, 96\}$, $\{1, 3, 6, 18, 72\}$, $\{1, 3, 6, 30, 60\}$, $\{1, 3, 12, 84\}$, or $\{1, 3, 24, 72\}$, which implies the longest SG sequence has a length of 5, hence $\sigma(100) = 5$. □

Problem 5 (4 points). Find, with proof, the value of $\sigma(360)$ and all SG sequences with length $\sigma(360)$ that sum to 360.

Solution. The first term of the SG sequence must divide 360, so it must be either 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, or 360.
If it's 360, it has a length of 1.
If it's 180, it cannot form an SG sequence.
If it's 120, the only possible sequence is $\{120, 240\}$, which has a length of 2.
If it's 90, the only possible SG sequence is $\{90, 270\}$, which has a length of 2.
If it's 72, the only possible SG sequence is $\{72, 288\}$, which has a length of 2.
If it's 60, the only possible SG sequence is $\{60, 300\}$, which has a length of 2.
If it's 45, the only possible SG sequence is $\{45, 315\}$, which has a length of 2.
If it's 40, then the remaining sequence can either be $\{320\}$ or $\{80, 240\}$, which implies the longest sequence has a length of 3.
If it's 36, then the remaining sequence can either be $\{324\}$ or $\{108, 216\}$, which implies the longest sequence has a length of 3.
If it's 30, the only possible SG sequence is $\{30, 330\}$, which has a length of 2.
If it's 24, then the remaining sequence can either be $\{48, 288\}$ or $\{48, 96, 192\}$, which implies the longest sequence has a length of 4.
If it's 20, the only possible SG sequence is $\{20, 340\}$, which has a length of 2.
If it's 18, the only possible SG sequence is $\{18, 342\}$, which has a length of 2.
If it's 15, the only possible SG sequence is $\{15, 345\}$, which has a length of 2.
If it's 12, the only possible SG sequence is $\{12, 348\}$, which has a length of 2.

If it's 10, then the remaining sequence can either be $\{350\}$, $\{50, 300\}$, $\{50, 100, 200\}$, or $\{70, 280\}$, which implies the longest sequence has a length of 4.
If it's 9, then the remaining sequence can either be $\{351\}$, $\{27, 324\}$, $\{27, 54, 270\}$, $\{27, 81, 243\}$, $\{27, 108, 216\}$, or $\{117, 234\}$, which implies the longest sequence has a length of 4.
If it's 8, the the remaining sequence can either be $\{352\}$, $\{16, 336\}$, $\{16, 48, 288\}$, $\{16, 48, 96, 192\}$, $\{16, 112, 224\}$, $\{32, 320\}$, $\{32, 64, 256\}$, or $\{88, 264\}$, which implies the longest sequence has a length of 5.
If it's 6, the only possible SG sequence is $\{6, 354\}$, which has a length of 2.
If it's 5, the only possible SG sequence is $\{5, 355\}$, which has a length of 2.
If it's 4, the only possible SG seqeunce is $\{4, 356\}$, which has a length of 2.
If it's 3, then the remaining sequence can either be $\{21, 336\}$, $\{21, 42, 294\}$, $\{21, 84, 252\}$, $\{51, 306\}$, or $\{51, 102, 204\}$, which implies the longest sequence has a length of 4.
If it's 2, the only possible SG sequence is $\{2, 358\}$, which has a length of 2.
If it's 1, the only possible SG sequecne is $\{1, 359\}$, which has a length of 2.
Hence $\sigma(360) = 5$ and the sequence is $\{8, 16, 48, 96, 192\}$. □

3 Properties of SG Numbers

In this section, we will prove some general properties of SG numbers.

Problem 6 (2 points). Prove that there exists a positive integer n such that there are five distinct SG sequences with length $\sigma(n)$ that sum to n.

Solution. Consider $n = 25$. The first term of the SG sequence must divide 25, so it must be either 1, 5, or 25.
If it's 25, it has a length of 1.
If it's 5, the only possible sequence is $\{5, 20\}$, which has a length of 2.
If it's 1, then the remaining sequence can either be $\{24\}$, $\{2, 22\}$, $\{3, 21\}$, $\{4, 20\}$, $\{6, 18\}$, or $\{8, 16\}$, which implies the longest sequence has a length of 3.
Hence, when $n = 25$, there are 5 distinct sequences with length $\sigma(n)$. □

Problem 7 (3 points). For any positive integer k, what is the smallest integer n such that $\sigma(n) = k$?

Solution. Since $\sigma(n) = k$, we have the SG sequence to be $a_1, a_1k_1, a_1k_1k_2, \cdots, a_1 \times \prod_{i=1}^{n-1} k_i$. Hence, $n = a_1 + a_1k_1 + a_1k_1k_2 + \cdots + a_1 \times \prod_{i=1}^{n-1} k_i$. We have $a_1 \geq 1$ and $k_i \geq 2$, so

$$n \geq 1 + 2 + 4 + \cdots + 2^{k-1} = 2^k - 1.$$

□

Problem 8 (3 points). If a positive integer n has k digits in its binary (base-2) representation with $k-1$ ones and 1 zero, prove that $\sigma(n) = k - 1$.

Solution. Let the digit with zero be m, we can have

$$n = (\prod_{i=0}^{m-2} 2^i) + (\prod_{i=m}^{k-1} 2^i).$$

Since $2^{i+1} = 2 \cdot 2^i$ and $2^m = 4 \cdot 2^{m-2}$, we have n to be the sum of a SG sequence with length $k - 1$, so $\sigma(n) \geq k - 1$.

Also, since the smallest integer n' such that $\sigma(n') = k$ is $2^k - 1$, which is greater than n, so $\sigma(n) < k$. Thus, $\sigma(n) = k - 1$. □

Problem 9 (4 points). Find, with proof, the second **and** the third smallest integers a, b such that $\sigma(a) = \sigma(b) = 10$.

Solution. The least integer n such that $\sigma(n) = 10$ is proved to be $2^{10} - 1 = 1023$, with the SG sequence to be $\{1, 2, 4, 8, 16, 32, 64, 128, 256, 512\}$. Since we want the second and the third least values, we have to change k_i. Because k_i are all 2 in the case with least value, we should change one k_i to 3. If we change k_i to 3, all values of a after k_i will increase, so we should change k_9 to 3. In this way, the sequence becomes $\{1, 2, 4, 8, 16, 32, 64, 128, 256, 768\}$, which sums up to 1279.
Similarly, to find the third least value, it could either change k_8 to 3 or change k_9 to 4. Changing k_8 to 3 results in a sum of 1407 and changing k_9 to 4 results in a sum of 1535, so the third least value is 1407. □

Problem 10

(a) Prove that for all integers $a, b > 1$, $\sigma(ab + 1) > \sigma(a)$.

Solution. Let $\sigma(a) = n$, we have

$$a = a_1 + a_2 + \cdots + a_n.$$

Then, we have

$$ab + 1 = 1 + a_1 b + a_2 b + \cdots + a_n b$$

Since $a_{i+1} = k_i a_i$, we have $a_{i+1} b = k_i(a_i b)$, so $\{1, a_1 b, a_2 b, \cdots, a_n b\}$ is a valid SG sequence with length $n + 1$. Hence, $\sigma(ab + 1) > n$. □

(b) Prove that for all integers $a, b > 1$, $\sigma(ab) \geq \sigma(a)$. Provide an example where the equality holds.

Solution. Similarly, let $\sigma(a) = n$, we have

$$a = a_1 + a_2 + \cdots + a_n.$$

Then, we have

$$ab = a_1 b + a_2 b + \cdots + a_n b$$

Since $a_{i+1} = k_i a_i$, we have $a_{i+1} b = k_i(a_i b)$, so $\{a_1 b, a_2 b, \cdots, a_n b\}$ is a valid SG sequence with length n. Hence, $\sigma(ab) \geq n$.
When $a = 3$, $b = 2$, we have $\sigma(a) = \sigma(ab) = 2$. □

Problem 11 (5 points). Find the largest integer n such that $\sigma(n) = 2$, and prove that any integer greater than n must have SG number of greater than 2.

Solution. Let the integer greater than 24 be n.
Consider the case when n is odd, then $n = 2k + 1$. For $k \geq 3$, $\sigma(k) \geq 2$ since $k = 1 + (k - 1)$. Then, since $\sigma(2k + 1) > \sigma(k)$, we have $\sigma(2k + 1) > 2$.

Consider the case when n is even.
Consider $n = 16k$, we have

$$n = k + 3k + 12k$$

so $\sigma(n) \geq 3$. Then n is not divisible by 16.

Consider $n = 8k$ where k is odd.
Let $k = 2a + 1$. Since when $a \geq 3$, $\sigma(a) \geq 2$, so $\sigma(2a + 1) > \sigma(a) \geq 2$. Hence, $\sigma(k) > 3$ for $k \geq 7$.
When $k = 5$, we have $n = 40$. Since $40 = 1 + 3 + 6 + 30 = 1 + 3 + 9 + 27$, so $\sigma(40) = 4 > 2$.
Therefore, $\sigma(n) \geq 3$.

Consider $n = 8k + 2$ where $k \geq 3$, we have

$$n = 2 + 2k + 6k$$

so $\sigma(n) \geq 3$.

Consider $n = 8k + 4$ where $k \geq 3$, we have

$$n = 4 + 8 + 8(k - 1)$$

so $\sigma(n) \geq 3$.

Consider $n = 8k + 6$ where $k \geq 3$, we have

$$8k + 6 = 2 + 4 + 8k$$

so $\sigma(n) \geq 3$.

Hence, any integer greater than 24 must have SG number of greater than 2. □

The following two problems are only worth one point each. This is not because the following problems are necessarily any easier than the others. In fact it's rather the opposite. We strongly advise that the students work on previous problems first and make sure they have everything correct before they dive into the last two.

Problem 12 (1 point). Prove that for all $n \geq 3$, the number $24n$ has SG number of greater than 3.

Solution. Consider $n = 3k$ where $k \geq 1$, we have

$$24n = 72k = 2k + 10k + 20k + 40k$$

so $\sigma(24n) \geq 4$.

Consider $n = 3k + 1$ where $k \geq 1$, we have

$$24n = 72k + 24 = 6 + 18 + 18k + 54k$$

so $\sigma(24n) \geq 4$ for $k \geq 2$. When $k = 1$, we have $24n = 96 = 1 + 5 + 10 + 20 + 60$. Hence, $\sigma(24n) \geq 4$.

Consider $n = 3k + 2$ where $k \geq 2$, we have

$$24n = 72k + 48 = 3 + 9 + 36 + 72k$$

so $\sigma(24n) \geq 4$.

Hence, we have $\sigma(24n) > 3$ for all $n \geq 3$. □

Problem 13 (1 point). Prove that for all $n > 48$ such that 24 does not divide n, $\sigma(n) > 3$.

Solution. Consider the case when n is odd, then $n = 2k + 1$ where $k \geq 24$.
When $k = 24$, $n = 49 = 1 + 3 + 9 + 36$, so $\sigma(49) = 4 > 3$.
When $k > 24$, we have $\sigma(k) \geq 3$, so $\sigma(2k + 1) > \sigma(k) \geq 3$. Then $\sigma(2k + 1) > 3$.

Consider the case when n is even.
Consider $n = 12k$ where k is odd, then $k = 2a + 1$ where $a \geq 2$. Hence, $n = 24a + 12$, so we have

$$n = 4 + 8 + 24 + 24(a - 1)$$

so $\sigma(n) \geq 4$ for $a > 2$. When $a = 2$, we have $n = 60 = 4 + 8 + 16 + 32$. Hence, $\sigma(n) \geq 4$.

Consider $n = 12k + 2$ where $k \geq 4$.
Consider the case when k is odd, then $k = 2a + 1$, so we have

$$n = 24a + 14 = 2 + 4 + 8 + 24a$$

so $\sigma(n) \geq 4$.
Consider the case when k is even, then $k = 2a$ where $a \geq 2$.
Consider the case when a is odd, then $a = 2b + 1$ where $b \geq 1$, so we have

$$n = 48b + 26 = 2 + 8 + 16 + 48b$$

so $\sigma(n) \geq 4$.
Consider the case when a is even, then $a = 2b$ where $b \geq 1$.
Consider $b = 3c$ where $c \geq 1$, we have

$$n = 144c + 2 = 2 + 4c + 20c + 120c$$

so $\sigma(n) \geq 4$.
Consider $b = 3c + 1$ where $c \geq 0$, we have

$$n = 144c + 50 = 2 + 12 + 36 + 144c$$

so $\sigma(n) \geq 4$ for $c \geq 1$. When $c = 0$, we have $n = 50 = 1 + 7 + 14 + 28$. Hence, $\sigma(n) \geq 4$.
Consider $b = 3c + 2$ where $c \geq 0$, we have

$$n = 144c + 98 = 2 + 6 + 18 + 72 + 144c$$

so $\sigma(n) \geq 4$.
Hence, we have $\sigma(n) \geq 4$.

Consider $n = 12k + 4$ where $k \geq 4$, we have

$$12k + 4 = 1 + 3 + 12 + 12(k - 1)$$

so $\sigma(n) \geq 4$.

Consider $n = 12k + 6$ where $k \geq 4$, we have

$$12k + 6 = 2 + 4 + 12 + 12(k - 1)$$

so $\sigma(n) \geq 4$.

Consider $n = 12k + 8$ where $k \geq 4$, we have

$$12k + 8 = 2 + 6 + 12 + 12(k - 1)$$

so $\sigma(n) \geq 4$.

Consider $12k + 10$ where $k \geq 4$, we have

$$12k + 10 = 1 + 3 + 6 + 12k$$

so $\sigma(n) \geq 4$.

Hence, we have $\sigma(n) > 3$ for all $n > 48$ such that 24 does not divide n. $\square$