Team Round

Duke Math Meet 2018

November 2018

Problem 1. If $f(x) = 3x - 1$, what is $f^6(2) = (f \circ f \circ f \circ f \circ f \circ f)(2)$?

Problem 2. A frog starts at the origin of the (x, y) plane and wants to go to $(6, 6)$. It can either jump to the right one unit or jump up one unit. How many ways are there for the frog to jump from the origin to $(6, 6)$ without passing through point $(2, 3)$?

Problem 3. Alfred, Bob, and Carl plan to meet at a café between noon and 2 pm. Alfred and Bob will arrive at a random time between noon and 2pm. They will wait for 20 minutes or until 2pm for all 3 people to show up after which they will leave. Carl will arrive at the café at noon and leave at 1:30 pm. What is the probability that all three will meet together?

Problem 4. Let triangle ABC be isosceles with $AB = AC$. Let BD be the altitude from B to AC, E be the midpoint of AB, and AF be the altitude from A to BC. If $AF = 8$ and the area of triangle ACE is 8, find the length of CD.

Problem 5. Find the sum of the unique prime factors of

$$(2018^2 - 121) \cdot (2018^2 - 9).$$

Problem 6. Compute the remainder when

$$3^{102} + 3^{101} + \cdots + 3^0$$

is divided by 101.

Problem 7. Take regular heptagon $DUKMATH$ with side length 3. Find the value of

$$\frac{1}{DK} + \frac{1}{DM}.$$

Problem 8. RJ's favorite number is a positive integer less than 1000. It has final digit of 3 when written in base 5 and final digit 4 when written in base 6. How many guesses do you need to be certain that you can guess RJ's favorite number?

Problem 9. Let $f(a, b) = \frac{a^2+b^2}{ab-1}$, where a and b are positive integers, $ab \neq 1$. Let x be the maximum positive integer value of f, and let y be the minimum positive integer value of f. What is $x - y$?

Problem 10. Haoyang has a circular cylinder container with height 50 and radius 5 that contains 5 tennis balls, each with outer-radius 5 and thickness 1. Since Haoyang is very smart, he figures out that he can fit in more balls if he cuts each of the balls in half, then puts them in the container, so he is "stacking" the halves. How many balls would he have to cut up to fill up the container?