2019 Duke Math Meet

Problems and Solutions

Saturday 2^{nd} November, 2019

1 Individual Problems

Problem 1.1. Compute the value of N, where

$$N = 818^3 - 6 \cdot 818^2 \cdot 209 + 12 \cdot 818 \cdot 209^2 - 8 \cdot 209^3.$$

Solution. $\boxed{64,000,000.}$

Note that N is simply the binomial expansion of

$$(818 - 2 \cdot 209)^3,$$

so

$$\begin{aligned} N &= (818 - 2 \cdot 209)^3 \\ &= 400^3 = 64,000,000. \end{aligned}$$

□

Problem 1.2. Suppose $x \leq 2019$ is a positive integer that is divisible by 2 and 5, but not 3. If 7 is one of the digits in x, how many possible values of x are there?

Solution. $\boxed{27.}$

If x is divisible by 2 and 5, it must be divisible by 10, thus the last digit must be 0, so there are only 3 variable digits of x. We now do casework on the number of digits of x.

Case 1: 2 digits. x must include the digit 7, so $x = 70$, which is not divisible by 3. Thus, there is 1 2-digit possibilities for x.

Case 2: 3 digits. Since x must include the digit 7, we either have $7a0$ or $a70$, and since x is not divisible by 3, we have a total of $7 + 6 - 1 = 12$ 3-digit possibilities for x.

Case 3: 4 digits. Similar to the previous case, we either have $x = 7ab0$, $a7b0$, or $ab70$. But, since $x \leq 2019$, we must have $x = 1a70$ or $17a0$. Thus, we get a total of $7 + 7 = 14$ 4-digit possibilities for x.

Summing these gives $1 + 12 + 14 = 27$. □

Problem 1.3. Find all non-negative integer solutions (a, b) to the equation

$$b^2 + b + 1 = a^2.$$

Solution. $\boxed{(1, 0).}$

We have

$$b^2 < b^2 + b + 1 = a^2 \leq b^2 + 2b + 1 = (b + 1)^2.$$

Hence, the only way for a to be an integer is if $a = b + 1$. This implies that $(a, b) = (1, 0)$ is the only solution. $\square$

Problem 1.4. Compute the remainder when $\sum_{n=1}^{2019} n^4$ is divided by 53.

Solution. $\boxed{25.}$

Notice that $\sum_{x=1}^{n} x^4$ is equivalent to counting the number of 5-tuples $(x_1, x_2, x_3, x_4, x_5)$ from $\{1, 2, \ldots, n + 1\}$ with $x_5 > \max(x_1, x_2, x_3, x_4)$ (you can see this by doing casework on x_5). The number of 5-tuples is also equal to

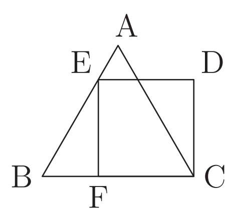
$$\binom{n+1}{2} + 14\binom{n+1}{3} + 36\binom{n+1}{4} + 24\binom{n+1}{5}.$$

This shows that $\sum_{x=1}^{n} x^4$ is divisible by n. Thus, since $2019 = 53 \cdot 38 + 5$, we can reduce our desired sum to

$$1^4 + 2^4 + 3^4 + 4^4 + 5^5 \mod 5,$$

which we can easily compute to be 25. $\square$

Problem 1.5. Let ABC be an equilateral triangle and $CDEF$ a square such that E lies on segment AB and F on segment BC. If the perimeter of the square is equal to 4, what is the area of triangle ABC?



Solution. $\boxed{\frac{1}{2} + \frac{\sqrt{3}}{3}.}$

We easily find $EF = 1$, so since $\angle B = 60°$, we know $BF = \frac{1}{\sqrt{3}}$, so $BC = 1 + \frac{1}{\sqrt{3}}$. Therefore, the area of ABC is

$$\frac{\sqrt{3}}{4}\left(1 + \frac{1}{\sqrt{3}}\right)^2 = \frac{1}{2} + \frac{\sqrt{3}}{3}.$$

$\square$

Problem 1.6.

$$S = \frac{4}{1\times 2\times 3} + \frac{5}{2\times 3\times 4} + \frac{6}{3\times 4\times 5} + \cdots + \frac{101}{98\times 99\times 100},$$

Let $T = \frac{5}{4} - S$. If $T = \frac{m}{n}$, where m and n are relatively prime integers, find the value of $m+n$.

Solution. $\boxed{6667.}$

$$\begin{aligned}
S &= \frac{4}{1\times 2\times 3} + \frac{5}{2\times 3\times 4} + \frac{6}{3\times 4\times 5} + ... + \frac{101}{98\times 99\times 100} \\
&= (\frac{3}{1\times 2\times 3} + \frac{4}{2\times 3\times 4} + \frac{5}{3\times 4\times 5} + ... + \frac{100}{98\times 99\times 100}) \\
&\quad + (\frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \frac{1}{3\times 4\times 5} + ... + \frac{1}{98\times 99\times 100}) \\
&= (\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ... + \frac{1}{98\times 99}) \\
&\quad + (\frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \frac{1}{3\times 4\times 5} + ... + \frac{1}{98\times 99\times 100}) \\
&= (\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{98} - \frac{1}{99}) \\
&\quad + \frac{1}{2}(\frac{1}{1\times 2} - \frac{1}{2\times 3} + \frac{1}{2\times 3} - \frac{1}{3\times 4} + \ldots + \frac{1}{98\times 99} - \frac{1}{99\times 100}) \\
&= (1 - \frac{1}{99}) + \frac{1}{2}(\frac{1}{2} - \frac{1}{9900}) \\
&= \frac{5}{4} - (\frac{1}{99} + \frac{1}{19800}) \\
&= \frac{5}{4} - \frac{67}{6600}.
\end{aligned}$$

Thus, $\frac{m}{n} = \frac{67}{6600}$, giving the answer of 6667. □

Problem 1.7. Find the sum of

$$\sum_{i=0}^{2019} \frac{2^i}{2^i + 2^{2019-i}}.$$

Solution. $\boxed{1010.}$

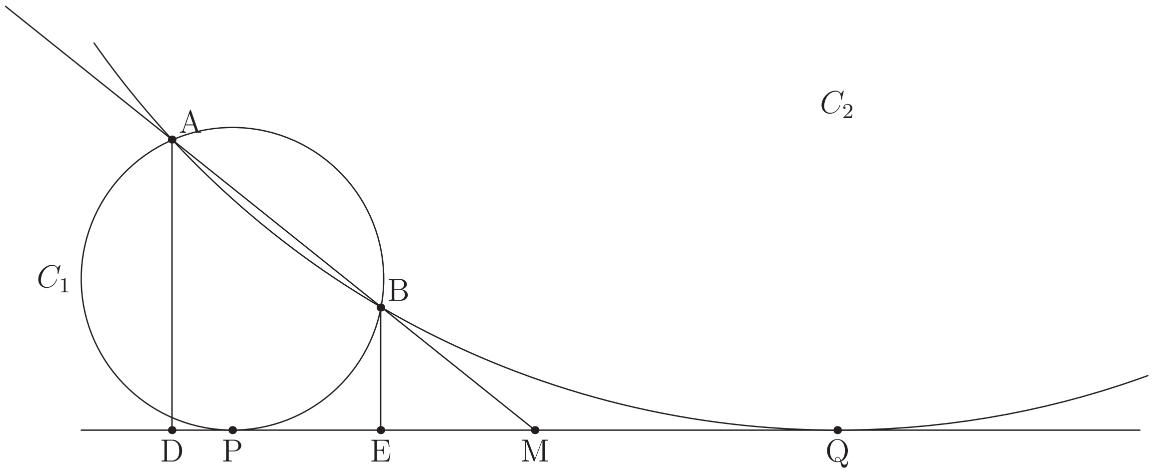
Note that

$$\frac{2^i}{2^i + 2^{2019-i}} + \frac{2^{2019-i}}{2^i + 2^{2019-i}} = 1,$$

so we can pair all the elements in the sum to get 1010. □

Problem 1.8. Let A and B be two points in the Cartesian plane such that A lies on the line $y = 12$, and B lies on the line $y = 3$. Let C_1, C_2 be two distinct circles that intersect both A and B and are tangent to the x-axis at P and Q, respectively. If $PQ = 420$, determine the length of AB.

Solution. $\boxed{315.}$



Draw the line through A and B, and let it hit the x-axis at M. The power of M with respect to C_1 is $PM^2 = BM \cdot AM$, and the power of M with respect to C_2 is $MQ^2 = MB \cdot MA$, so we find that $PM = MQ = 210$. Let D be the perpendicular from A to the x-axis and let E be the perpendicular from B to the x-axis. Then, similar triangles MBE and MAD give us $\frac{BM}{AM} = \frac{3}{12} = \frac{1}{4}$, so combining this with $BM \cdot AM = 210^2$ gives $BM = 105$, so $AM = 420$, and $AB = 315$. □

Problem 1.9. Zion has an average 2 out of 3 hit rate for 2-pointers and 1 out of 3 hit rate for 3-pointers. In a recent basketball match, Zion scored 18 points without missing a shot, and all the points came from 2 or 3-pointers. What is the probability that all his shots were 3-pointers?

Solution. $\boxed{\frac{27}{8435}.}$

The conditional probability tells us that the probability is the probability of shooting all 3-pointers over the probability of scoring 18 points without missing, which is

$$\frac{\left(\frac{1}{3}\right)^6}{\left(\frac{1}{3}\right)^6 + \left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^3\binom{7}{4} + \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^6\binom{8}{2} + \left(\frac{2}{3}\right)^9} = \frac{27}{8435}.$$

□

Problem 1.10. Let $S = \{1, 2, 3, \ldots, 2019\}$. Find the number of non-constant functions $f : S \to S$ such that

$$f(k) = f(f(k+1)) \leq f(k+1) \quad \text{for all } 1 \leq k \leq 2018.$$

Express your answer in the form $\binom{m}{n}$, where m and n are integers.

Solution. $\boxed{\binom{2019}{3}}$.

Suppose $f(1) = a$. Then, $f(f(2)) = a \leq f(2)$. If $f(2) > a$, then $f(f(2)) \geq f(a) \geq f(2) > a$, which is a contradiction by the monotonicity of f. Therefore, $f(2) = a$. By a similar argument, we see that $f(a+1) = a$. Now, suppose $f(i) = b$, with $f(j) = a$ with $a < b$ for all $j < i$. Then, $f(f(i+1)) = b \leq f(i+1)$. If $f(i+1) = b$, then $f(f(i+1)) \geq f(b) = a$ because $b < i$, which makes a contradiction. On the other hand, if $f(i+1) > b$, then there is no other value $x \in S$ such that $f(x) = b$, so $x = i$, hence $f(i+1) = i$. Therefore, we see that the number of non-constant functions satisfying the given condition is the same as the number of ways to pick a, b, and i, where $a < b < i$. This is simply the number of ways to pick 3 distinct numbers from S, or $\binom{2019}{3}$. □