4 Relay Problems

4.1 Problem 1

Problem 4.1.1. We can write 2019 as

$$2019 = a^4 + b^4 + c^4 + d^4 + e^4,$$

where a, b, c, d, and e are integers. Find the minimum value of $|a+b+c+d+e|$, where $|x|$ is the absolute value function.

Solution. $\boxed{1.}$

We can easily find $2019 = 1^4 + 2^4 + 3^4 + 5^4 + 6^4$. There are 3 odd integers and 2 even integers, so we know the minimum value of $|a+b+c+d+e|$ must be odd. We find that $(a, b, c, d, e) = (1, 2, -3, 5, -6)$ gives $|a+b+c+d+e| = 1$, so the minimum value is 1. □

Problem 4.1.2. Let T = TNYWR, and let $ABCDEFGH$ be a regular octagon centered at O with $AO = 4T$. Determine the area of the incircle of $ACEG$.

Solution. $\boxed{8\pi.}$

Before receiving the value of T, notice that the side length of $ACEG$ is $AC = \sqrt{2}AO = 4\sqrt{2}T$, since AOC is a 45-45-90 triangle. Then, the radius of the incircle is $\frac{4\sqrt{2}T}{2} = 2\sqrt{2}T$, so the area of the incircle is $8T^2\pi$. Since $T = 1$, the answer is 8π. □

Problem 4.1.3. Let T = TNYWR, and let $N = \frac{T}{\pi}$. Suppose a bag of 15 apples has N rotten apples. What is the probability that if I randomly pick apples from the bag without replacement, the 11^{th} apple I draw is the last rotten one? Express your answer as a common fraction.

Solution. $\boxed{\dfrac{8}{429}.}$

Before receiving the value of T, notice that the probability that the 11^{th} apple is the last rotten one is equivalent to the the probability of a sequence of 15 letters, N of which are "R" and $15-N$ of which are "C" (for rotten and clean), has the last R on the 11^{th} slot. This probability is equal to number of ways to order 10 letters with $N-1$ labeled "R", divided by the total number of ways to order the 15 letters. Thus, it is equal to

$$\frac{\binom{10}{N-1}}{\binom{15}{N}}.$$

Since $T = 8\pi$, $N = 8$, so our answer is

$$\frac{\binom{10}{7}}{\binom{15}{8}} = \frac{8}{429}.$$

□

4.2 Problem 2

Problem 4.2.1. A cafeteria has 4 entrees and 5 desserts. How many different meals can Jung eat if he eats either 1 or 2 entrees and either 1 or 2 desserts?

Solution. $\boxed{150.}$

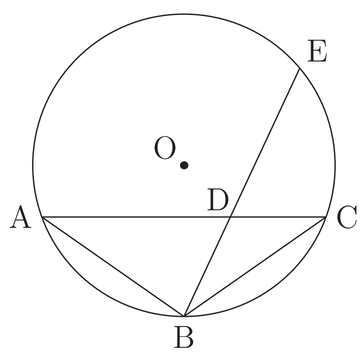
He has $\left(\binom{4}{1}+\binom{4}{2}\right)\left(\binom{5}{1}+\binom{5}{2}\right)=150$ options. □

Problem 4.2.2. Let T = TNYWR. Find the total number of positive integers $n \leq T$ such that n^4+5n^2+9 is not divisible by 5.

Solution. $\boxed{30.}$

Before receiving T: notice that $n^4+5n^2+9=(n-1)(n+1)(n^2+1)+5(n^2+2)$, so if $n \equiv \pm 1 \mod 5$, then either $5|(n-1)$ or $5|(n+1)$, and if $n \equiv \pm 2 \mod 5$, then $5|(n^2+1)$. However, if $5|n$, then n^4+5n^2+9 is not divisible by 5, so the answer is the number of multiples of 5 less than or equal ot T. $T=150$, so the answer is $\frac{150}{5}=30$. □

Problem 4.2.3. Let T = TNYWR. In the diagram below, let $AB=BC=\frac{T}{2}$, and $BD=\frac{T}{3}$. Find the length of DE.



Solution. $\boxed{\frac{25}{2}.}$

Let $AD=m$ and $DC=n$. Before receiving the value of T, we apply Stewart's theorem on $\triangle ABC$ to find that

$$\begin{aligned} AD\cdot AC\cdot DC+AC\cdot BD^2 &= DC\cdot AB^2+AD\cdot BC^2 \\ (m+n)\cdot(mn+\tfrac{T^2}{9}) &= \tfrac{T^2}{4}(m+n) \\ mn+\tfrac{T^2}{9} &= \tfrac{T^2}{4} \\ mn &= \tfrac{5T^2}{36}. \end{aligned}$$

By Power of a Point on D, we see that $AD\cdot DC=BD\cdot DE$, so

$$DE=\frac{AD\cdot DC}{BD}=\frac{mn}{T/3}=\frac{5T^2/36}{T/3}=\frac{5T}{12}.$$

Since $T=30$, we get $DE=\frac{150}{12}=\frac{25}{2}$. □