# 2 Tiebreakers

**Problem 2.1.** Let $a(1), a(2), \ldots, a(n), \ldots$ be an increasing sequence of positive integers satisfying $a(a(n)) = 3n$ for every positive integer $n$. Compute $a(2019)$.

*Solution.* $\boxed{3870.}$

If $a(1) = 1$ we also have $a(a(1)) = 1 \neq 3 \cdot 1$ which is impossible. Since the sequence is increasing, it follows that $1 < a(1) < a(a(1)) = 3$ and thus $a(1) = 2$. From the equation we deduce $a(3n) = a(a(a(n))) = 3a(n)$ for all $n$. We easily prove by induction (starting with $a(1) = 2$) that $a(3^m) = 2 \cdot 3^m$ for every $m$. Using this we also obtain $a(2 \cdot 3^m) = a(a(3^m)) = 3^{m+1}$.

There are $3^n - 1$ integers $i$ such that $3^n < i < 2 \cdot 3^n$ and there are $3^n - 1$ integers $j$ such that $a(3^n) = 2 \cdot 3^n < j < 3^{n+1} = a(2 \cdot 3^n)$. Since $a(n)$ is increasing there is no other option than $a(3^n + b) = 2 \cdot 3^n + b$ for all $0 < b < 3^n$. Therefore $a(2 \cdot 3^n + b) = a(a(3^n + b)) = 3^{n+1} + 3b$ for all $0 < b < 3^n$. Since $2019 = 2 \cdot 3^6 + 561$, we have $a(2019) = 37 + 3 \cdot 561 = 3870$. □

**Problem 2.2.** Consider the function $f(12x - 7) = 18x^3 - 5x + 1$. Then, $f(x)$ can be expressed as $f(x) = ax^3 + bx^2 + cx + d$, for some real numbers $a$, $b$, $c$ and $d$. Find the value of $(a + c)(b + d)$.

*Solution.* $\boxed{\frac{135}{64}.}$

The problem asks for a product of some sums of coefficients of $f(x)$, suggesting that values like $f(1)$ and $f(-1)$ are useful in finding the target. We know that $f(1) = a+b+c+d$ and $f(-1) = -a+b-c+d$, so we have that $f(1)+f(-1) = 2(b+d)$ and $f(1)-f(-1) = 2(a+c)$, meaning that

$$(f(1) + f(-1))(f(1) - f(-1)) = 4(a + c)(b + d) = f^2(1) - f^2(-1),$$

by difference of squares. But we know from the given equation that $f(1)$ is obtained when $x = \frac{2}{3}$ and $f(-1)$ when $x = \frac{1}{2}$ , so plugging in, we have that

$$f(1) = 18\frac{2}{3}^3 - 5\frac{2}{3} + 1 = 3$$

and

$$f(-1) = 18\frac{1}{2}^3 - 5\frac{1}{2} + 1 = \frac{3}{4}.$$

Therefore, $f^2(1) - f^2(-1) = 9 - \frac{9}{16} = \frac{135}{16}$, so $(a + c)(b + d) = \frac{135}{64}$. □

**Problem 2.3.** Let $a$, $b$ be real numbers such that $\sqrt{5+2\sqrt{6}} = \sqrt{a} + \sqrt{b}$. Find the largest value of the quantity

$$X = \frac{1}{a + \frac{1}{b + \frac{1}{a + \cdots}}}.$$

*Solution.* $\boxed{\dfrac{-3+\sqrt{15}}{2}.}$

We can easily find $\sqrt{5+2\sqrt{6}} = \sqrt{2} + \sqrt{3}$. So, either $a = 2$ and $b = 3$ or $a = 3$ and $b = 2$. Note also that

$$X = \frac{1}{a + \frac{1}{b+X}}.$$

In order to maximize $X$, we must minimize the denominator, or $a + \frac{1}{b+X}$. This is obviously minimized with $a = 2$, so the expression becomes

$$X = \frac{1}{2 + \frac{1}{3+X}},$$

which we can solve to find $X = \frac{-3+\sqrt{15}}{2}$. □