1 Individual Problems

Problem 1.1. Four witches are riding their brooms around a circle with circumference 10m. They are standing at the same spot, and then they all start to ride clockwise with the speed of $1, 2, 3$, and 4 m/s, respectively. Assume that they stop at the time when every pair of witches has met for at least two times (the first position before they start counts as one time). What is the total distance all the four witches have travelled?

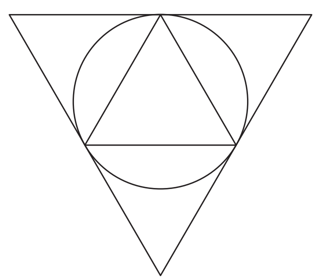
Solution. $\boxed{100.}$

We can see that they will stop when the witches with speed 3 and 4 meet for the second time. If they meet the second time after s seconds, then $4s = 3s + 10$, so $s = 10$. Then, the total distance traveled is $10 \cdot (1 + 2 + 3 + 4) = 100$.

□

Problem 1.2. Suppose A is an equilateral triangle, O is its inscribed circle, and B is another equilateral triangle inscribed in O. Denote the area of triangle T as $[T]$. Evaluate $\frac{[A]}{[B]}$.

Solution. $\boxed{4.}$



Suppose A has side length a. Since O is the inscribed circle of A, the radius r of O is $\frac{a}{2}/\sqrt{3} = \frac{a}{2\sqrt{3}}$. Since B is an equilateral triangle inscribed in O, its side length b satisfies $b = \sqrt{3}r$. Hence, $b = \frac{a}{2}$, so $\frac{[A]}{[B]} = 2^2 = 4$. □

Problem 1.3. Tim has bought a lot of candies for Halloween, but unfortunately, he forgot the exact number of candies he has. He only remembers that it's an even number less than 2020. As Tim tries to put the candies into his unlimited supply of boxes, he finds that there will be 1 candy left if he puts seven in each box, 6 left if he puts eleven in each box, and 3 left if he puts thirteen in each box. Given the above information, find the total number of candies Tim has bought.

Solution. $\boxed{666}$

Let x be the total number of candies that Tim has bought. Then, we have:

$$\begin{aligned} x &\equiv 0 \mod 2, \\ x &\equiv 1 \mod 7, \\ x &\equiv 6 \mod 11, \\ x &\equiv 3 \mod 13. \end{aligned}$$

From the last two, we must have $x \equiv 3 + 7 \cdot 13 \equiv 6 + 8 \cdot 11 = 94 \mod 143$. Combining with the first equation gives us $x \equiv 94 \mod 286$, and combining with the second equation gives us $x \equiv 94 + 2 \cdot 286 = 1 + 95 \cdot 7 \equiv 666 \mod (2 \cdot 7 \cdot 11 \cdot 13 = 1502)$. Then, since 666 is the only integer less than 2020 that is congruent to $666 \mod 2002$, we have $x = 666$. □

Problem 1.4. Let $f(n)$ be a function defined on positive integers n such that $f(1) = 0$, and $f(p) = 1$ for all prime numbers p, and

$$f(mn) = nf(m) + mf(n)$$

for all positive integers m and n. Let

$$n = 277945762500 = 2^2 3^3 5^5 7^7.$$

Compute the value of $\frac{f(n)}{n}$.

Solution. $\boxed{4.}$

Let us consider the general case, where $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. Let $S = \sum_{i=1}^{k} e_i$. We claim that $f(n)$ is the sum of all factors α of n such the sum of the exponents in the prime factorization of α is equal to $S-1$. For example, $f(2^2 \cdot 3 \cdot 5) = 2\cdot2\cdot3+2\cdot2\cdot5+2\cdot3\cdot5+2\cdot3\cdot5$. We leave the induction proof as an exercise to the reader. Then, we can rewrite $f(n)$ as

$$f(n) = \underbrace{\frac{n}{p_1} + \cdots + \frac{n}{p_1}}_{e_1 \text{ times}} + \cdots + \underbrace{\frac{n}{p_k} + \cdots + \frac{n}{p_k}}_{e_k \text{ times}} = n \sum_{i=1}^{k} \frac{e_i}{p_i}.$$

Therefore, in this case, we have $f(n) = n(\frac{2}{2} + \frac{3}{3} + \frac{5}{5} + \frac{7}{7}) = 4n$, so $\frac{f(n)}{n} = 4$.

□

Problem 1.5. Compute the only positive integer value of $\frac{404}{r^2-4}$, where r is a rational number.

Solution. $\boxed{2500.}$

Let $r = \frac{a}{b}$, where a and b are relatively prime integers. Then, we have

$$\frac{404}{r^2-4} = \frac{404}{\frac{a^2}{b^2}-4} = \frac{404b^2}{a^2-4b^2} = \frac{404b^2}{(a-2b)(a+2b)}.$$

Since the greatest common factor of a, b is 1, we know that the greatest common factor of b and $a-2b$ as well as $a+2b$ is also 1. Therefore, both $a-2b$ and $a+2b$ must be factors of 404 in order for the fraction to be an integer, and no factors from either term in the denominator may be drawn from b^2.

Since $404 = 2^2 \cdot 101$, we can check all possibilities for the values of $a-2b$ and $a+2b$. In particular, we check cases where $(a-2b)(a+2b) = 4, 101$, and 404, and solve for integers a and b. Then, we see that the only solution that works is $a-2b = 1$ and $a+2b = 101$, giving us $a = 51$ and $b = 25$, so $r = \frac{51}{25}$. Then, plugging this back into the original expression gives us

$$\frac{404}{r^2-4} = 404 \cdot \frac{625}{101} = 2500.$$

□

Problem 1.6. Let $\alpha = 3 + \sqrt{10}$.. If

$$\prod_{k=1}^{\infty}\left(1 + \frac{5\alpha+1}{\alpha^k+\alpha}\right) = m + \sqrt{n},$$

where m and n are integers, find $10m + n$.

Solution. $\boxed{50.}$

The key observation here is $\alpha^2 = 6\alpha + 1$. Using this fact, we can simplify the expression:

$$\begin{aligned}\prod_{k=1}^{\infty}\left(1 + \frac{5\alpha+1}{\alpha^k+\alpha}\right) &= \prod_{k=1}^{\infty}\left(\frac{\alpha^k+6\alpha+1}{\alpha^k+\alpha}\right)\\ &= \prod_{k=1}^{\infty}\left(\frac{\alpha^k+\alpha^2}{\alpha^k+\alpha}\right)\\ &= \prod_{k=0}^{\infty}\left(\frac{\alpha^k+\alpha}{\alpha^k+1}\right).\end{aligned}$$

The product of the first $n+1$ terms of this product is

$$(1+\alpha)\left(\frac{\alpha^n}{\alpha^n+1}\right),$$

and as n grows, the fraction grows infinitely close to 1, so the product is equal to

$$\alpha + 1 = 4 + \sqrt{10}.$$

Therefore, $10m + n = 40 + 10 = 50$. □

Problem 1.7. Charlie is watching a spider in the center of a hexagonal web of side length 4. The web also consists of threads that form equilateral triangles of side length 1 that perfectly tile the hexagon. Each minute, the spider moves unit distance along one thread. If $\frac{m}{n}$ is the probability, in lowest terms, that after four minutes the spider is either at the edge of her web or in the center, find the value of $m+n$.

Solution. $\boxed{241.}$

We note that at each move the spider either moves closer to the edge, maintains its distance, or moves away from the edge. For the spider to reach the edge, it needs to move forwards four times. Not all forward moves are the same, as some forwards moves allows the spider to go to three possible forward moves while some forward moves only allow the spider to go to two possible forward moves. We will call these moves f_3 and f_2 for convenience. We see that by doing some casework that

$$\begin{aligned}\mathbb{P}(f \to f_3 \to f_3 \to f_3) &= 1 \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{72},\\ \mathbb{P}(f \to f_3 \to f_2 \to f_2) &= 1 \cdot \frac{1}{6} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{54},\\ \mathbb{P}(f \to f_2 \to f_2 \to f_2) &= 1 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}.\end{aligned}$$

Now, we calculate the probability that the spider ends up back at the center. It can do so by moving back twice (after being forced to move forwards). Or moving to the side twice then moving back. We have that

$$\begin{aligned}\mathbb{P}(f \to b \to f \to b) &= 1 \cdot \frac{1}{6} \cdot 1 \cdot \frac{1}{6} = \frac{1}{36},\\ \mathbb{P}(f \to s \to s \to b) &= 1 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{54}.\end{aligned}$$

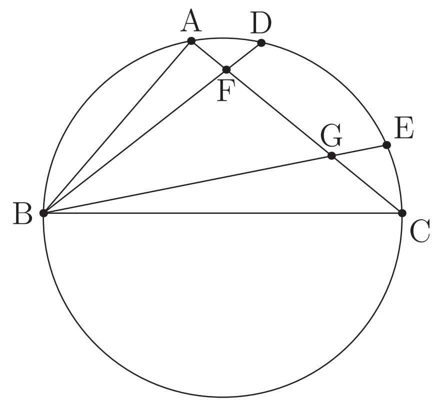
Adding all these cases together gives us the total probability of $\frac{25}{216}$, so the answer is $25+216 = 241$. □

Problem 1.8. Let ABC be a triangle with $AB = 10, AC = 12$, and ω its circumcircle. Let F and G be points on $\overline{AC}$ such that $AF = 2$, $FG = 6$, and $GC = 4$, and let $\overrightarrow{BF}$ and $\overrightarrow{BG}$ intersect ω at D and E, respectively. Given that AC and DE are parallel, what is the square of the length of BC?

Solution. $\boxed{250.}$

Denote $x = BC$. Since $ACED$ is an isosceles trapezoid, we may put $y = AE = CD$. Finally, let $p = BF$, $q = DF$, $u = BG$, and $v = GE$. Note that $\angle BAC$ and $\angle BDC$ are inscribed in the same circle, so they have the same measure. Therefore, $\triangle ABF$ and $\triangle DCF$ are similar, so

$$\frac{DF}{AF} = \frac{CD}{AB} = \frac{CF}{BF} \implies \frac{q}{2} = \frac{y}{10} = \frac{10}{p}.$$



Similarly (pun intended), we have that $\triangle BCG$ and $\triangle AEG$ are similar, so we have

$$\frac{AE}{BC} = \frac{EG}{CG} = \frac{AG}{BG} \implies \frac{y}{x} = \frac{v}{4} = \frac{8}{u}.$$

Lastly, since $AC \| DE$, we have

$$\frac{p}{q} = \frac{u}{v},$$

so combining all of the above gives us

$$\frac{p}{q} = \frac{\frac{100}{y}}{\frac{y}{5}} = \frac{\frac{8x}{y}}{\frac{4y}{x}},$$

so $500 = 2x^2$, and $x^2 = 250$. □

Problem 1.9. Two blue devils and 4 angels go trick-or-treating. They randomly split up into 3 non-empty groups. Let p be the probability that in at least one of these groups, the number of angels is nonzero and no more than the number of devils in that group. If $p = \frac{m}{n}$ in lowest terms, compute $m + n$.

Solution. $\boxed{76.}$

There are three ways to partition 6 into 3 groups: $(4, 1, 1), (3, 2, 1)$, and $(2, 2, 2)$. In the first case, there are a total of $\binom{6}{2} = 15$ ways to make the groups. To satisfy the criteria, the two devils must be in the group of 4, hence $\binom{4}{2} = 6$ groupings. In the second case, there are a total of $6 \cdot \binom{5}{2} = 60$ ways to make the groups. To satisfy the criteria, either the two devils are in the group of 3, or there is exactly one devil in the group of 2. There are $4 \cdot \binom{3}{2} + 2 \cdot 4 \cdot \binom{4}{1} = 44$ groupings. In the last case, there are $\frac{\binom{6}{2}\binom{4}{2}}{3!} = 15$ total ways to make the groups. To satisfy the criteria, the two devils cannot be in the same group, giving us $\frac{\binom{4}{2}}{2!} = 3$ bad groupings, so 12 groups that work. This gives us a total probability of $\frac{6+44+12}{15+60+15} = \frac{62}{90} = \frac{31}{45}$, so the answer is $31 + 45 = 76$.

□

Problem 1.10. We know that

$$2^{22000} = \underbrace{4569878\ldots229376}_{6623 \text{ digits}}.$$

For how many positive integers $n < 22000$ is it also true that the first digit of 2^n is 4?

Solution. $\boxed{2132}$

If the first digit of a k-digit number N is c, then $c10^k \leq N < (c+1)10^{k-1}$. This implies that $2c10^{k-1} \leq 2N < (2c+2)10^{k-1}$, i.e. the first digit of $2N$ is at least the first digit of $2c$ and at most the first digit of $2c+1$. We apply this to the first digits of powers of two: Having a power of two with the first digit equal to 1, there are these five possibilities for the first digits of the following powers of two: (1) 1,2,4,8,1; (2) 1,2,4,9,1; (3) 1,2,5,1; (4) 1,3,6,1; (5) 1,3,7,1.

Let k be a non-negative integer such that $2k$ begins with 1 and has d digits. Then, there is a unique power of two beginning with 1 and having $d+1$ digits, and it is either 2^{k+3} (if we are in one of the situations (3), (4), (5) above) or 2^{k+4} (given that the case (1) or (2) occurs). As 2^0 (having 1 digit) and 2^{21998} (having 6623 digits) begin with 1, we can compute how many times (1) or (2) occurs when computing successive powers of two: It is exactly $21998 - 3 \cdot 6622 = 2132$ times.

Finally, observe that the case (1) and (2) are precisely those giving rise to a power of two starting with 4, therefore there are exactly 2132 such numbers in the given range.

□