2 Team Problems

Problem 2.1. At Duke, $\frac{1}{2}$ of the students like lacrosse, $\frac{3}{4}$ like football, and $\frac{7}{8}$ like basketball. Let p be the proportion of students who like at least all three of these sports and let q be the difference between the maximum and minimum possible values of p. If q is written as $\frac{m}{n}$ in lowest terms, find the value of $m + n$.

Problem 2.2. A *dukie* word is a 10-letter word, each letter is one of the four D, U, K, E such that there are four consecutive letters in that word forming the letter $DUKE$ in this order. For example, $DUDKDUKEEK$ is a dukie word, but $DUEDKUKEDE$ is not. How many different dukie words can we construct in total?

Problem 2.3. Rectangle $ABCD$ has sides $AB = 8$, $BC = 6$. $\triangle AEC$ is an isosceles right triangle with hypotenuse AC and E above AC. $\triangle BFD$ is an isosceles right triangle with hypotenuse BD and F below BD. Find the area of $BCFE$.

Problem 2.4. Chris is playing with 6 pumpkins. He decides to cut each pumpkin in half horizontally into a top half and a bottom half. He then pairs each top-half pumpkin with a bottom-half pumpkin, so that he ends up having six "recombinant pumpkins". In how many ways can he pair them so that only one of the six top-half pumpkins is paired with its original bottom-half pumpkin?

Problem 2.5. Matt comes to a pumpkin farm to pick 3 pumpkins. He picks the pumpkins randomly from a total of 30 pumpkins. Every pumpkin weighs an integer value between 7 to 16 (including 7 and 16) pounds, and there're 3 pumpkins for each integer weight between 7 to 16. Matt hopes the weight of the 3 pumpkins he picks to form the length of the sides of a triangle. Let $\frac{m}{n}$ be the probability, in lowewst terms, that Matt will get what he hopes for. Find the value of $m + n$

Problem 2.6. Let a, b, c, d be distinct complex numbers such that $|a| = |b| = |c| = |d| = 3$ and $|a + b + c + d| = 8$. Find $|abc + abd + acd + bcd|$.

Problem 2.7. A board contains the integers $1, 2, \ldots, 10$. Anna repeatedly erases two numbers a and b and replaces it with $a + b$, gaining $ab(a + b)$ lollipops in the process. She stops when there is only one number left in the board. Assuming Anna uses the best strategy to get the maximum number of lollipops, how many lollipops will she have?

Problem 2.8. Ajay and Joey are playing a card game. Ajay has cards labelled $2, 4, 6, 8$, and 10, and Joey has cards labelled $1, 3, 5, 7, 9$. Each of them takes a hand of 4 random cards and picks one to play. If one of the cards is at least twice as big as the other, whoever played the smaller card wins. Otherwise, the larger card wins. Ajay and Joey have big brains, so they play perfectly. If $\frac{m}{n}$ is the probability, in lowest terms, that Joey wins, find $m + n$.

Problem 2.9. Let $ABCDEFGHI$ be a regular nonagon with circumcircle ω and center O. Let M be the midpoint of the shorter arc AB of ω, P be the midpoint of MO, and N be the midpoint of BC. Let lines OC and PN intersect at Q. Find the measure of $\angle NQC$ in degrees.

Problem 2.10. In a 30×30 square table, every square contains either a kit-kat or an oreo. Let T be the number of triples (s_1, s_2, s_3) of squares such that s_1 and s_2 are in the same row, and s_2 and s_3 are in the same column, with s_1 and s_3 containing kit-kats and s_2 containing an oreo. Find the maximum value of T.