Problem 2.1. At Duke, $\frac{1}{2}$ of the students like lacrosse, $\frac{3}{4}$ like football, and $\frac{7}{8}$ like basketball. Let p be the proportion of students who like at least all three of these sports and let q be the difference between the maximum and minimum possible values of p. If q is written as $\frac{m}{n}$ in lowest terms, find the value of $m+n$.

Solution. $\boxed{11.}$

The maximum occurs when the $\frac{1}{2}$ that like lacrosse alsoo like football and basketball, so the maximum is $\frac{1}{2}$. To find the minimum, note that the minimum amount that like both lacrosse and football is $\frac{1}{4}$, so we want the minimal overlap between this $\frac{1}{4}$ and the $\frac{7}{8}$ basketball lovers, which is $\frac{1}{8}$ of the student population. Thus, $q = 1 - \frac{1}{8} = \frac{3}{8}$, giving the final answer of 11.

□

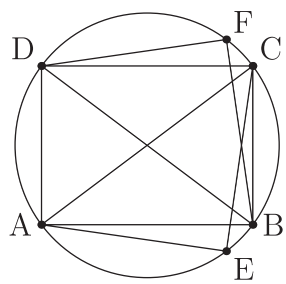
Problem 2.2. A *dukie* word is a 10-letter word, each letter is one of the four D, U, K, E such that there are four consecutive letters in that word forming the letter $DUKE$ in this order. For example, $DUDKDUKEEK$ is a dukie word, but $DUEDKUKEDE$ is not. How many different dukie words can we construct in total?

Solution. $\boxed{28576}$

First, we count the number of dukie words with at least one $DUKE$ present. We can see that there are $10-4+1=7$ possible positions for the word $DUKE$, and for the remaining 6 positions, there are 4^6 ways to choose the letter, so there are $7 \cdot 4^6$ dukie words with at least one $DUKE$ present. Now we count the number of dukie words with two $DUKE$ presences. We can treat the word $DUKE$ as one "super letter", so for a word with 2 $DUKE$s present, there are 2 remaining positions, each of which have 4 letter choices. Then, we have two letters and two super letters, giving us $4^2 \cdot \binom{4}{2}$ dukie words with 2 $DUKE$s present. Thus, the total number of dukie words is $7 \cdot 4^6 - 4^2 \cdot \binom{4}{2} = 28576$. □

Problem 2.3. Rectangle $ABCD$ has sides $AB = 8$, $BC = 6$. $\triangle AEC$ is an isosceles right triangle with hypotenuse AC and E above AC. $\triangle BFD$ is an isosceles right triangle with hypotenuse BD and F below BD. Find the area of $BCFE$.

Solution. $\boxed{7}$



Apply Ptolemy's Theorem on $AEBC$ to get $(10)(EB)+(5\sqrt{2})(6) = (5\sqrt{2})(8)$, so $EB = \sqrt{2}$. Applying Ptolemy's again on $EBCF$ gives us $(EF)(6)+(\sqrt{(2)})^2 = (5\sqrt{2})^2$, so $EF = 8$. Since $EBCF$ is isosceles, the distance from E to AB is 1, so by the Pythagorean Theorem, the height is 1. The area is therefore $\frac{6+8}{2} \cdot 1 = 7$. □

Problem 2.4. Chris is playing with 6 pumpkins. He decides to cut each pumpkin in half horizontally into a top half and a bottom half. He then pairs each top-half pumpkin with a bottom-half pumpkin, so that he ends up having six "recombinant pumpkins". In how many ways can he pair them so that only one of the six top-half pumpkins is paired with its original bottom-half pumpkin?

Solution. $\boxed{264.}$

There are 6 ways to choose which of the 6 pumpkins is restored correctly. The other five are deranged (all halves paired incorrectly). If D_n denotes the number of derangements for n pairs of objects, we know that $D_n = (n-1)(D_{n-1} + D_{n-2})$, where $D_1 = 0$ and $D_2 = 1$ (the proof of this is left as an exercise to the reader). Then, we have $D_5 = 44$, so there are $6 \cdot 44 = 264$ ways to pair the pumpkins so that only one of the pumpkins is correctly restored. □

Problem 2.5. Matt comes to a pumpkin farm to pick 3 pumpkins. He picks the pumpkins randomly from a total of 30 pumpkins. Every pumpkin weighs an integer value between 7 to 16 (including 7 and 16) pounds, and there're 3 pumpkins for each integer weight between 7 to 16. Matt hopes the weight of the 3 pumpkins he picks to form the length of the sides of a triangle. Let $\frac{m}{n}$ be the probability, in lowewst terms, that Matt will get what he hopes for. Find the value of $m + n$

Solution. $\boxed{8003.}$

We compute the complement: the three weights do not form a triangle. The triplets for which this happens are: $(7, 7, 14), (7, 7, 15), (7, 7, 16), (7, 8, 15), (7, 8, 16), (7, 9, 16), (8, 8, 16)$. For the triplets of the form (a, a, b), there are $\binom{3}{2} \cdot 3 = 9$ combinations of the pumpkins, and for the triplets of the form (a, b, c), there are $3^3 = 27$ combinations of the pumpkins. Therefore, the complement is $9 \cdot 4 + 27 \cdot 3 = 117$, so the desired probability is

$$1 - \frac{117}{\binom{30}{3}} = \frac{3943}{4060}.$$

Hence, the answer is $3943 + 4060 = 8003$.

□

Problem 2.6. Let a, b, c, d be distinct complex numbers such that $|a| = |b| = |c| = |d| = 3$ and $|a + b + c + d| = 8$. Find $|abc + abd + acd + bcd|$.

Solution. $\boxed{72.}$

Note that

$$|abc + abd + acd + bcd| = |abcd| \left| \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right|,$$

since magnitudes are distributive over multiplication. The trick is to express $\frac{1}{z}$ as $\frac{\overline{z}}{|z|^2}$, and to note that $|z| = |\overline{z}|$. Then, we have:

$$\begin{aligned} |abcd| \left| \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right| &= |a||b||c||d| \left| \frac{\overline{a}}{|a|^2} + \frac{\overline{b}}{|b|^2} + \frac{\overline{c}}{|c|^2} + \frac{\overline{d}}{|d|^2} \right| \\ &= 3^4 \left| \frac{\overline{a}}{9} + \frac{\overline{b}}{9} + \frac{\overline{c}}{9} + \frac{\overline{d}}{9} \right| \\ &= 9|\overline{a} + \overline{b} + \overline{c} + \overline{d}| \\ &= 9 \left| \overline{a + b + c + d} \right| \\ &= 9|(a + b + c + d)| = 9 \cdot 8 = 72. \end{aligned}$$

□

Problem 2.7. A board contains the integers $1, 2, \ldots, 10$. Anna repeatedly erases two numbers a and b and replaces it with $a + b$, gaining $ab(a + b)$ lollipops in the process. She stops when there is only one number left in the board. Assuming Anna uses the best strategy to get the maximum number of lollipops, how many lollipops will she have?

Solution. $\boxed{54450.}$

After replacing a and b with $a+b$, Anna will gain $ab(a+b) = \frac{(a+b)^3-a^3-b^3}{3}$ lollipops. Therefore, when the numbers $a+b$ and c and replaced with $a+b+c$, Anna will gain $\frac{(a+b+c)^3-(a+b)^3-c^3}{3}$, and combined with the first quantity, results in an overall net gain of $\frac{(a+b+c)^3-a^3-b^3-c^3}{3}$. Thus, we can see that at the end, Anna will have

$$\frac{(1+2+\cdots+10)^3-1^3-2^3-\cdots-10^3}{3} = 54450$$

lollipops.

□

Problem 2.8. Ajay and Joey are playing a card game. Ajay has cards labelled $2, 4, 6, 8$, and 10, and Joey has cards labelled $1, 3, 5, 7, 9$. Each of them takes a hand of 4 random cards and picks one to play. If one of the cards is at least twice as big as the other, whoever played the smaller card wins. Otherwise, the larger card wins. Ajay and Joey have big brains, so they play perfectly. If $\frac{m}{n}$ is the probability, in lowest terms, that Joey wins, find $m+n$.

Solution. $\boxed{19.}$

First note that 1 beats everything, so if Joey has it in his hand, then he will always play it and win. Thus, we just need to consider the case when Joey doesn't draw the 1. Also note that because of this, Ajay will play assuming Joey doesn't draw the 1, because it is the only way that Ajay can win.

Note that 3 beats every card except the 4, while 9 beats only beats a 6 and 8, so playing the 9 is strictly worse than playing the 3. Thus, Joey will never play the 9, and Ajay knows this, so Ajay will play assuming Joey will play one of 3, 5, or 7.

Now, we look at Ajay's options. Both 2 and 8 beat 5 and 7, while 6 and 10 only beat 5 and 7, respectively. Thus, Ajay will never play 6 and 10, since they are strictly worse than both 2 and 8. We can further simplify by noticing that 5 and 7 are equivalent for Joey, since both beat 4 but lose to 2 and 8, and 2 and 8 are equivalent for Ajay.

Thus, if we let p be the probability that Ajay chooses 4 when he has a 4 in his hand, we have that overall the probability of him playing 4 is $.8 \cdot p$, so the probability of playing 2 or 8 is $1-0.8p$. To ensure that Joey doesn't gain an advantage, these two must be equal, so we set $0.8p = 1-0.8p$, or $p = \frac{5}{8}$, and to ensure that Ajay doesn't gain an advantage, Joey picks 3 with probability $\frac{1}{2}$ and 5 or 7 with probability $\frac{1}{2}$. Therefore, Joey will win with probability

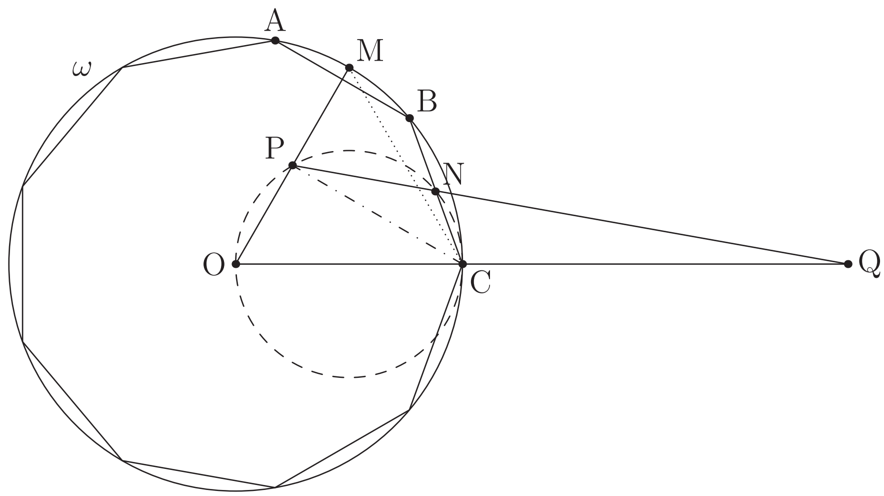
$$\frac{4}{5} + \frac{1}{5} \cdot \frac{1}{2} = \frac{9}{10},$$

so our final answer is $9+10 = 19$.

□

Problem 2.9. Let $ABCDEFGHI$ be a regular nonagon with circumcircle ω and center O. Let M be the midpoint of the shorter arc AB of ω, P be the midpoint of MO, and N be the midpoint of BC. Let lines OC and PN intersect at Q. Find the measure of $\angle NQC$ in degrees.

Solution. $\boxed{10.}$



Since C and M lie on ω, we have $OC = OM$, and $\angle MOC = \angle MOB + \angle BOC = 20 + 40 = 60$, so $\triangle OCM$ is equilateral. Then, since P is the midpoint of OM, we have $\angle OPC = 90^\circ$. Since $\angle ONC = 90^\circ$ because N is the midpoint of BC, we have that quadrilateral $OCNP$ is cyclic. Furthermore, $\angle OCN = 180^\circ - 20^\circ - 90^\circ = 70^\circ$, so $\angle OPN = 180^\circ - \angle OCN = 110^\circ$ because $OCNP$ is cyclic. Therefore, using $\triangle OQP$, we have

$$\angle NQC = \angle PQO = 180^\circ - \angle POQ - \angle QPO = 10^\circ.$$

□

Problem 2.10. In a 30×30 square table, every square contains either a kit-kat or an oreo. Let T be the number of triples (s_1, s_2, s_3) of squares such that s_1 and s_2 are in the same row, and s_2 and s_3 are in the same column, with s_1 and s_3 containing kit-kats and s_2 containing an oreo. Find the maximum value of T.

Solution. $\boxed{120000.}$

We claim that in an $n \times n$ square table there are at most $\frac{4n^2}{27}$ such triples.

Let row i and column j contain a_i and b_j kit-kats respectively, and let R be the set of red cells. For every red cell (i, j) there are $a_i b_j$ admissible triples (C_1, C_2, C_3) with $C_2 = (i, j)$, therefore

$$T = \sum_{(i,j)\in R} a_i b_j.$$

We use the inequality $2ab \le a^2 + b^2$ to obtain

$$T \le \frac{1}{2} \sum_{(i,j)\in R} (a_i^2 + b_j^2) = \frac{1}{2} \sum_{i=1}^{n} (n - a_i) a_i^2 + \frac{1}{2} \sum_{j=1}^{n} (n - b_j) b_j^2.$$

This is because there are $n - a_i$ red cells in row i and $n - b_j$ red cells in column j. Now we maximize the right-hand side.

By the AM-GM inequality we have

$$(n-x)x^2 = \frac{1}{2}(2n-2x)\cdot x \cdot x \leq \frac{1}{2}\left(\frac{2n}{3}\right)^3 = \frac{4n^3}{27},$$

with equality if and only if $x = \frac{2n}{3}$. By putting everything together, we get

$$T \leq \frac{n}{2}\frac{4n^3}{27} + \frac{n}{2}\frac{4n^3}{27} = \frac{4n^4}{27}.$$

If $n = 30$, then any coloring of the square table with $x = \frac{2n}{3} = 20$ kit-kats in each row and column attains the maximum as all inequalities in the previous argument become equalities. For example, let a cell (i, j) contain a kit-kat if i − j ≡ 1, 2, . . . , 20 (mod 30), and red otherwise.

Therefore the maximum value T can attain is $T = \frac{4 \cdot 30^4}{27} = 120000$. □