3 Team

1. In basketball, teams can score $1, 2$, or 3 points each time. Suppose that Duke basketball have scored 8 points so far. What is the total number of possible ways (ordered) that they have scored? For example, $(1, 2, 2, 2, 1), (1, 1, 2, 2, 2)$ are two different ways.

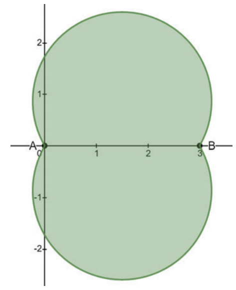
   Solution: $\boxed{81}$. We can use the recurrence $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ where a_n denote the number of sequences for n points. Since $a_1 = 1, a_2 = 2, a_3 = 4$, the next few numbers are $7, 13, 24, 44, 81$.

2. All the positive integers that are coprime to 2021 are grouped in increasing order, such that the nth group contains $2n - 1$ numbers. Hence the first three groups are $\{1\}$, $\{2, 3, 4\}$, $\{5, 6, 7, 8, 9\}$. Suppose that 2022 belongs to the kth group. Find k.

   Solution: $\boxed{44}$. From $2021 = 43 \times 47$, there are 1933 numbers up to 2022. The first n groups will include a total of n^2 numbers, and so $k = \lceil \sqrt{1933} \rceil = 44$.

3. Let $A = (0, 0)$ and $B = (3, 0)$ be points in the Cartesian plane. If R is the set of all points X such that $\angle AXB \geq 60^\circ$ (all angles are between 0° and 180°), find the integer that is closest to the area of R.

   Solution: $\boxed{15}$. Assume that X is in the positive y half-plane. Consider the circle with center in the positive y half-plane that passes through A and B such that the arc AB has angle 120°. Clearly, any point X satisfies the desired inequality iff it is within the circle. To handle points X in the negative y half-plane, we can reflect our work across the x-axis. Thus, R looks like the following:



From here, we can use basic geometry tricks to find that the area is $4\pi + \frac{3}{2}\sqrt{3} \approx 15.2$.

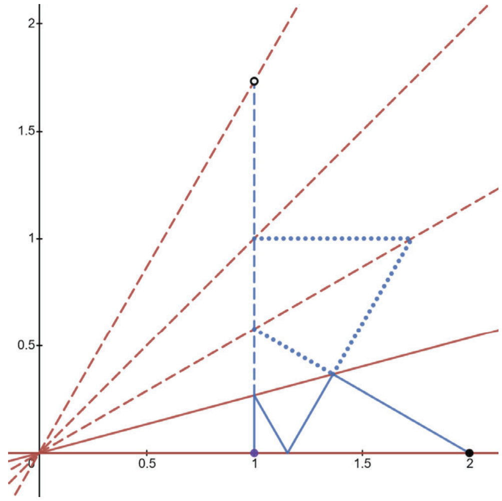
4. What is the smallest positive integer greater than 9 such that when its left-most digit is erased, the resulting number is one twenty-ninth of the original number?

   Solution: $\boxed{725}$.

   Let d be the first digit of the number, k be the number obtained after erasing the first digit, and n be the number of digits of k. Then the original number is $10^n d + k$, and the assertion can be rewritten as
   $$10^n d + k = 29k,$$
   or
   $$28k = 10^n d.$$
   Since $28 = 2^2 \cdot 7$, in order for the right-hand side to be divisible by 28, the value of d must be 7, and we must also have $n \geq 2$. The choice of $n = 2$ (implying $k = 25$) gives the smallest possible number 725.

5. Jonathan is operating a projector in the cartesian plane. He sets up 2 infinitely long mirrors represented by the lines $y = \tan(15°)x$ and $y = 0$, and he places the projector at $(1, 0)$ pointed perpendicularly to the x-axis in the positive y direction. Jonathan furthermore places a screen on one of the mirrors such that light from the projector reflects off the mirrors a total of three times before hitting the screen. Suppose that the coordinates of the screen is (a, b). Find $10a^2 + 5b^2$.

   Solution: $\boxed{40}$. We wish to avoid computation as much as possible. The key insight is that, when the light from the projector hits a mirror, we can reflect the "world" instead of reflecting the light. This is conveyed in the following diagram:



The purple point is the projector, the black points are the screens with the open point as the image of the screen, the solid red lines are the mirrors, the dotted red lines are the images of the mirrors, the solid blue line is the path of the light, the dashed blue line is the image of the path of the light, and the dotted blue lines represent where the image of the light would travel if they were reflected. The image of the mirror containing the image of the screen can be represented by the line $y = \tan(60°)x$. Since the image of the light follows the line $x = 1$, we find that the image of the light hits the image of the screen at $(1, \sqrt{3})$. This point is a distance of 2 away from the origin. Since the screen is on the x-axis, the screen must be on the point $(2, 0)$.

6. Dr Kraines has a cube of size $5 \times 5 \times 5$, which is made from 5^3 unit cubes. He then decides to choose m unit cubes that have an outside face such that any two different cubes don't share a common vertex. What is the maximum value of m?

   Solution: $\boxed{26}$.

   The optimal construction for the $(2n+1)^3$ cube, is such that each face we are able to choose $(n+1)^2$ cubes, and counting together and omit the overlapping cubes, we get $6n^2 + 2$. So the answer is $6 \cdot 2^2 + 2 = 26$.

   To justify that, we can find a way to partition all the outer faces into disjoint $2\times2\times1$ blocks, $2 \times 2 \times 1$ blocks and single cubes. There are a total of $6n^2 + 2$ of them and each of them contains at most one selected cube.

7. Let $a_n = \tan^{-1}(n)$ for all positive integers n. Suppose that

$$\sum_{k=4}^{\infty} (-1)^{\lfloor \frac{k}{2} \rfloor + 1} \tan(2a_k)$$

   is equals to $\frac{a}{b}$, where a, b are relatively prime. Find $a + b$.

   Solution: $\boxed{19}$. By the tangent addition formula, we have $\tan(2a_k) = \tan(a_k + a_k) = \frac{\tan(a_k)+\tan(a_k)}{1-\tan(a_k)\tan(a_k)} = \frac{2k}{(1+k)(1-k)}$. We can use partial fractions to see that $\frac{2k}{(1+k)(1-k)} = -\frac{1}{k-1} - \frac{1}{k+1}$.

We also have that -1 when $k \equiv 0, 1 \pmod 4$ and 1 when $k \equiv 2, 3 \pmod 4$. Then, the summation telescopes to $(\frac{1}{3} + \frac{1}{4}) = \frac{7}{12}$.

8. Rishabh needs to settle some debts. He owes 90 people and he must pay $\$(101050 + n)$ to the nth person where $1 \leq n \leq 90$. Rishabh can withdraw from his account as many coins of values \$2021 and $\$x$ for some fixed positive integer x as is necessary to pay these debts. Find the sum of the four least values of x so that there exists a person to whom Rishabh is unable to pay the exact amount owed using coins.

   Answer: $\boxed{195}$.

   Solution a. Firstly, note that $101050 = 2021 \cdot 50$. So for Rishabh to be able to pay all of his debts, there must exist nonnegative integers a, b for each $1 \leq n \leq 90$ such that $2021 \cdot 50 + n = 2021a + xb$. Clearly, $a \leq 50$, so we can rewrite this equation as $2021(50 - a) \equiv -n \pmod x$. If $43 \mid x$ or $47 \mid x$, then the LHS is equivalent to 0 always, so setting $n = 1$ yields that Rishabh cannot pay his debts. We now suppose that $\gcd(x, 2021) = 1$. If $x \leq 51$, then it is commonly known that $2021(50 - a)$ can be equivalent to any number $\pmod x$. Thus, Rishabh can pay off his debts in this case. If $x \geq 52$, then $2021(50 - a)$ only takes on 51 values since $0 \leq a \leq 50$. But n spans $\min(x, 90) \geq 52$ values. Therefore, Rishabh is unable to pay off his debts. Our desired values then are $43 + 47 + 52 + 53 = 195$.

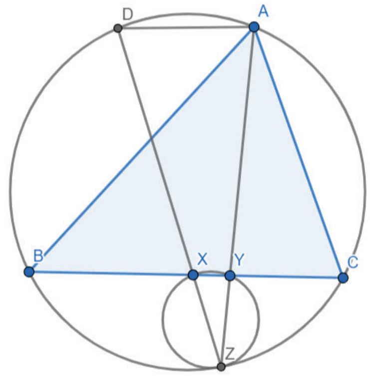
   Solution b. We proceed with the Chicken McNugget Theorem. We first note that if $43 \mid x$ or $47 \mid x$, then Rishabh can only pay in multiples of \$43 or \$47 respectively. This means he cannot pay off his debts. We then suppose that $\gcd(x, 2021) = 1$. The theorem posits that Rishabh can pay a given person if he owes them $\$(2021x - 2021 - x + 1)$ or more. If $x \leq 51$, then $(2021x - 2021 - x + 1) \leq 101050$ so Rishabh can pay off his debts. If $x \geq 52$, then we use a corollary that, for each integer k, exactly one of $\$k$ and $\$(2021x - 2021 - x - k)$ can be paid. For $x = 52$, we consider $k = 1924$. Since \$1924 can be paid, \$101095 cannot and Rishabh cannot pay his debts. For $x = 53$, we consider $k = 3922$ which implies that \$101117 cannot be paid. Rishabh cannot pay his debts in this case either. Since the problem asks only for the least 4 values, we can conclude that the desired values are $43 + 47 + 52 + 53 = 195$.

9. A frog starts at $(1, 1)$. Every second, if the frog is at point (x, y), it moves to $(x + 1, y)$ with probability $\frac{x}{x+y}$ and moves to $(x, y + 1)$ with probability $\frac{y}{x+y}$. The frog stops moving when its y coordinate is 10. Suppose the probability that when the frog stops its x-coordinate is strictly less than 16, is given by $\frac{m}{n}$ where m, n are positive integers that are relatively prime. Find $m + n$

   Solution: $\boxed{13}$. First show that the x-coordinate is k with probability exactly $\frac{9}{(9+k-1)(9+k)}$. This is because there are $\binom{9+k-2}{k}$ ways of reaching $(k, 10)$ as the last move must be $+1$ in y-coordinate, The probability of each path is exactly $\frac{9! \cdot (k-1)!}{(9+k)!}$. Thus the answer is $\sum_{k=1}^{15} \frac{9}{(k+8)(k+9)} = 9(\frac{1}{9} - \frac{1}{24}) = \frac{5}{8}$.

10. In the triangle ABC, $AB = 585$, $BC = 520$, $CA = 455$. Define X, Y to be points on the segment BC. Let $Z \neq A$ be the intersection of AY with the circumcircle of ABC. Suppose that XZ is parallel to AC and the circumcircle of XYZ is tangent to the circumcircle of ABC at Z. Find the length of XY.

Solution: $\boxed{64}$.



By cosine rule, we can first deduce that $\cos C = 2/7, \cos B = 2/3$. Now extend ZX to meet the circumcircle at D. By homothety, we deduce that $DACB$ is an isosceles trapezoid. Hence $ACXD$ is a parallelogram, and we have $DX = 455$. Using cosine rule, we can compute that $AD = CX = 260$, and so X is the midpoint of BC. By power of a point, $ZX = \frac{CX \cdot BX}{XD} = \frac{260^2}{455} = \frac{1040}{7}$. Finally, by $XYZ \sim DAZ$, we have $XY = AD \cdot \frac{ZX}{ZD} = \frac{260 \times 1040/7}{455 + 1040/7} = 64$.