5 Tiebreaker

1. You are standing on one of the faces of a cube. Each turn, you randomly choose another face that shares an edge with the face you are standing on with equal probability, and move to that face. Let $F(n)$ the probability that you land on the starting face after n turns. Supposed that $F(8) = \frac{43}{256}$, and $F(10)$ can be expressed as a simplified fraction $\frac{a}{b}$. Find $a+b$.

   Solution: $\boxed{1195}$

   Let's say that at the start you are standing on face 1 which is opposite of face 6, with the other faces being 2, 3, 4, 5. Define $G(n)$ to be the probability that you land on faces 2, 3, 4, or 5 after n turns. First we show that the probability you land on face 6 after n turns is also $F(n)$. This is clear because we can only reach face 6 from faces 2, 3, 4, 5, but you can reach face 1 from these faces with equal probability. It is also clear that $G(n) = \frac{1}{4}(1 - 2F(n))$ because all the probabilities must add up to 1. Furthermore, we can deduce that $F(n+1) = G(n)$. This is because to reach face 1 after $(n+1)$ turns, you must reach faces 2, 3, 4, or 5 after n turns (probability $= 4G(n)$), but from each face you have a $\frac{1}{4}$ probability of moving to 1 on the next step, so $F(n+1) = \frac{1}{4}4G(n) = G(n)$. From these observations, we can compute $F(9) = G(8) = \frac{1}{4}(1-2\times\frac{43}{256}) = \frac{85}{512}$, and $F(10) = G(9) = \frac{1}{4}(1-2\times\frac{85}{512}) = \frac{171}{1024}$, so $a+b = 1195$.