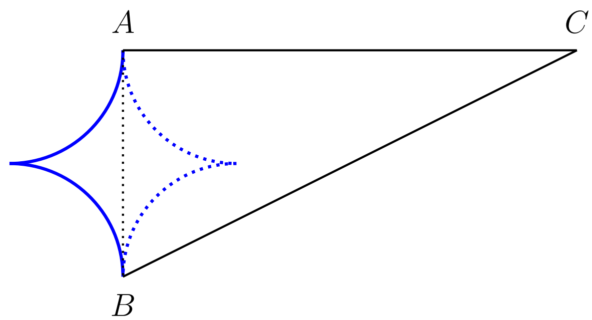
# Individual Round Solutions

DMM 2022

## 1 Individual Round

1. Sujay sees a shooting star go across the night sky, and took a picture of it. The shooting star consists of a star body, which is bounded by four quarter-circle arcs, and a triangular tail. Suppose $AB = 2$, $AC = 4$. Let the area of the shooting star be $X$. If $6X = a - b\pi$ for positive integers $a, b$, find $a + b$.




**Solution.** We have $6X = 6\left(4 + 2 - \dfrac{\pi}{2}\right) = 36 - 3\pi$. Hence, $a + b = \boxed{39}$.

2. Assuming that each distinct arrangement of the letters in DISCUSSIONS is equally likely to occur, what is the probability that a random arrangement of the letters in DISCUSSIONS has all the S's together?

**Solution.** The total number of arrangements is $\frac{11!}{2!4!}$, and the number of arrangements with S's together is $\frac{8!}{2!}$. Hence, the probability that a random arrangements has all the S's together is $\frac{8!/2!}{11!/(2!\cdot 4!)} = \boxed{\dfrac{4}{165}}$.

3. Evaluate
$$\frac{(1+2022)(1+2022^2)(1+2022^4)\cdots(1+2022^{2^{2022}})}{1+2022+2022^2+\ldots+2022^{2^{2023}-1}}.$$

**Solution.** Multiply both the numerator and the denominator by $(2022-1)$. Observe that $(x-1)(x+1)(x^2+1)\cdots(x^{2^n}+1) = x^{2^{n+1}} - 1$ and $(x-1)(1+x+x^2+\cdots+x^n) = x^{n+1}-1$, we get the answer is $\boxed{1}$.

4. Dr. Kraines has 27 unit cubes, each of which has one side painted red while the other five are white. If he assembles his cubes into one $3 \times 3 \times 3$ cube by placing each unit cube in a

random orientation, what is the probability that the entire surface of the cube will be white, with no red faces visible? If the answer is $2^a3^b5^c$ for integers $a, b, c$, find $|a + b + c|$.

**Solution.** Each of the 8 corner pieces has a $\frac{1}{2}$ probability of having only white faces visible, as 3 of the 6 faces are concealed by other unit cubes. The 12 edge pieces each have a $\frac{2}{3}$ probability of having only white faces visible, as 4 of the 6 possible locations for the red face to be are concealed. Finally, the 6 pieces at the center of a side have a $\frac{5}{6}$ chance of being placed so the 1 visible face is white. This gives a total probabilty of

$$\left(\frac{1}{2}\right)^8 \cdot \left(\frac{2}{3}\right)^{12} \cdot \left(\frac{5}{6}\right)^6 = \frac{5^6}{3^{18}2^2},$$

so the answer is $|6 - 18 - 2| = \boxed{14}$.

5. Let $S$ be a subset of $\{1, 2, 3, \ldots, 1000, 1001\}$ such that no two elements of $S$ have a difference of 4 or 7. What is the largest number of elements $S$ can have?

**Solution.** First consider taking such a subset of $\{1, 2, \ldots, 11\}$. There can be no disadvantage to including 1 in our subset, so we begin by including 1 and eliminating 5 and 8. Now, at most one from each pair $(2, 9)$, $(3, 7)$, $(4, 11)$, $(6, 10)$ can be chosen, so no more than 5 elements can be in our subset, with the subset $\{1, 3, 4, 6, 9\}$ being a possible example of a subset containing 5 elements.

Taking the complete set from the problem, it can be divided into 91 groups of 11, from each of which a maximum of 5 elements can be taken by the logic above. This gives an upper bound of $5 \cdot 91 = \boxed{455}$, which is possible by taking $S = \{x \mid x \in S,\ x \bmod 11 \in \{1, 3, 4, 6, 9\}\}$.

6. George writes the number 1. At each iteration, he removes the number $x$ written and instead writes either $4x + 1$ or $8x + 1$. He does this until $x > 1000$, after which the game ends. What is the minimum possible value of the last number George writes?

**Solution.** We will consider every number George writes in binary. In binary, turning $x$ into $4x + 1$ is equivalent to concatenating 01 right after $x$. Similarly, turning $x$ into $8x + 1$ is equivalent to concatenating 001 right after $x$. Essentially, George starts with 1, and at each step, he appends either 01 or 001 right after $x$.

Note that it is impossible to use only 10 digits. If we use only 10 digits, the second digit of $x$ must be 0, making it smaller than 1000. Thus, the final number must have at least 11 digits. The first digit of $x$ must be 1, and we want to fill the rest of the 10 digits with 01 and 001. Suppose we use $a$ 01's and $b$ 001's. Then, we have $2a + 3b = 10$, the only solution of which is $a = b = 2$. In order to minimize the value of $x$, we need to place 01's as right as possible and place 001's as left as possible. Thus, the binary representation of the final answer is 10010010101, which equals $\boxed{1173}$.

7. List all positive integer ordered pairs $(a, b)$ satisfying $a^4 + 4b^4 = 281 \cdot 61$.

**Solution.** By Sophie Germain, we can write

$$a^4 + 4b^4 = ((a+b)^2 + b^2)((a-b)^2 + b^2) = 61 \cdot 281.$$

Since $a, b > 0$, the first term is greater than the second term so we have two cases. If $(a-b)^2 + b^2 = 1$, then either $a - b = 0, b = 1$, or $a - b = 1, b = 0$, or $a - b = -1, b = 0$. None of them satisfies the given equation, so $(a-b)^2 + b^2 = 61$ and $(a+b)^2 + b^2 = 281$. The only two square numbers that sum to 61 are 25 and 36, so either $a - b = 5, b = 6$ or $a - b = 6, b = 5$ which implies that $a = 11, b = 5, 6$. Testing these two pairs on $(a+b)^2 + b^2 = 281$ gives $(a, b) = \boxed{(11, 5)}$.

8. Karthik the farmer is trying to protect his crops from a wildfire. Karthik's land is a $5 \times 6$ rectangle divided into 30 smaller square plots. The 5 plots on the left edge contain fire, the 5 plots on the right edge contain blueberry trees, and the other $5 \times 4$ plots of land contain banana bushes. Fire will repeatedly spread to all squares with bushes or trees that share a side with a square with fire. How many ways can Karthik replace 5 of his 20 plots of banana bushes with firebreaks so that fire will not consume any of his prized blueberry trees?

**Solution.** It is clear that each of the 5 rows must have a firebreak, and that the firebreaks must be at most 1 column away from adjacent firebreaks. Let the second and fifth columns be *edge* columns, and let the third and fourth columns be *center* columns. We proceed by recursion.

Let $E_k$ be the number of configurations where the firebreaks in the first $k$ rows are at most 1 columns away from each other and the $k$th firebreak is on an edge column. Likewise, let $C_k$ be the number of configurations where the firebreaks in the first $k$ rows are at most 1 columns away from each other and the $k$th firebreak is on a center column. We have the following recursive formula

$$E_k = E_{k-1} + C_{k-1}, C_k = E_{k-1} + 2C_{k-1}.$$

This is true because when we add a firebreak in the $k$th row and an edge column, the previous firebreak is either in an edge column or center column. When we add a firebreak in the $k$th row and a center column, the previous firebreak is either in an edge column or one of two center columns. Since $E_1 = C_1 = 2$, we can calculate $E_5$ and $C_5$ as follows:

| $k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $E_k$ | 2 | 4 | 10 | 26 | 68 |
| $C_k$ | 2 | 6 | 16 | 42 | 110 |

Thus, our answer is $68 + 110 = \boxed{178}$. Note that the terms in the table are actually twice the terms of the Fibonacci sequence which can be proven by induction.

9. Find $a_0 \in \mathbb{R}$ such that the sequence $\{a_n\}_{n=0}^{\infty}$ defined by $a_{n+1} = -3a_n + 2^n$ is strictly increasing.

**Solution.** Dividing both sides of the recurrence relation by $2^{n+1}$ gives

$$\frac{a_{n+1}}{2^{n+1}} = -\frac{3}{2} \cdot \frac{a_n}{2^n} + \frac{1}{2}.$$

Letting $b_n = \frac{a_n}{2^n}$ gives $b_{n+1} = -\frac{3}{2} b_n + \frac{1}{2}$, which is equivalent to $b_{n+1} - \frac{1}{5} = -\frac{3}{2}\left(b_n - \frac{1}{5}\right)$. Hence, we get $b_n - \frac{1}{5} = (b_0 - \frac{1}{5})(-\frac{3}{2})^n = (a_0 - \frac{1}{5})(-\frac{3}{2})^n$. Therefore, we have $a_n = 2^n \cdot b_n = (-3)^n[(a_0 - \frac{1}{5}) + \frac{1}{5}(-\frac{2}{3})^n]$. Since $\{a_n\}_{n=0}^{\infty}$ is strictly increasing, we must have $a_0 - \frac{1}{5} = 0$, *i.e.* $\boxed{a_0 = \frac{1}{5}}$.

10. Jonathan is playing with his life savings. He lines up a penny, nickel, dime, quarter, and half-dollar from left to right. At each step, Jonathan takes the leftmost coin at position 1 and uniformly chooses a position $2 \le k \le 5$. He then moves the coin to position $k$, shifting all coins at positions 2 through $k$ leftward. What is the expected number of steps it takes for the half-dollar to leave and subsequently return to position 5?

**Solution.** Let $T_k$ denote the expected number of steps required for a coin at position $k$ to reach position 1. Note that, at each step, the probability of the coin shifting from position $k$ to $k-1$ is $\frac{6-k}{4}$, so we expect that it will take $\frac{4}{6-k}$ steps to reach position $k-1$. This gives the recursive formula $T_k = \frac{4}{6-k} + T_{k-1}$ with $T_1 = 0$, so we can compute the following:

| $k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $T_k$ | 0 | 1 | $\frac{7}{3}$ | $\frac{13}{3}$ | $\frac{25}{3}$ |

The half-dollar will then take $\frac{25}{3}$ steps in expectation to reach position 1. Afterwards, there is a $\frac{1}{4}$ chance that the next step will return the half-dollar to position 5 and a $\frac{3}{4}$ chance that it will be taken elsewhere. This means that it takes $\frac{4}{1} - 1 = 3$ failed trials in expectation for the half-dollar to successfully return to position 5. Each failed trial is expected to take $1 + \frac{1}{3}(T_2 + T_3 + T_4)$ steps, and the successful trial takes only 1 step to return to position 5. Thus, the overall expected value is $T_5 + 3(1 + \frac{1}{3}(T_2 + T_3 + T_4)) + 1 = \boxed{20}$ steps.