Relay Round Solutions

DMM 2022

1 Relay Round

Problem 1

Problem 1.1: A robot is located at 2 on the number line, and it needs to reach either 5 or 0. Every second, there's a $\frac{1}{3}$ chance it breaks down, a $\frac{1}{3}$ chance it moves one unit in the positive direction, and a $\frac{1}{3}$ chance it moves one unit in the negative direction. The probability the robot manages to reach 5 or 0 before breaking down is $\frac{m}{n}$, where m and n are coprime. Find n.

Solution. Let P_n for $1 \le n \le 4$ be the probability that starting at point n, the robot will manage to reach 5 or 0 before breaking down. Then we have the system

$$\begin{aligned} P_1 &= \frac{1}{3} + \frac{1}{3}P_2, \\ P_2 &= \frac{1}{3}P_1 + \frac{1}{3}P_3, \\ P_3 &= \frac{1}{3}P_2 + \frac{1}{3}P_4, \\ P_4 &= \frac{1}{3}P_3 + \frac{1}{3}, \end{aligned}$$

solving which gives $P_2 = \frac{1}{5}$. Hence, $\boxed{n = 5}$.

Problem 1.2: Let $T =$ TNYWR. Navya, the fruit ninja, has a bitter feud with watermelon and strawberries. She can only cut 3 watermelon with one slice or T strawberries with one slice. Suppose she slices 17 times tomorrow, and let N be the total number of watermelon and strawberries she cuts tomorrow. How many possible values of N are prime?

Solution. Suppose Navya slices strawberries s times. Then, she cuts

$$3(17 - s) + 5s = 51 + 2s.$$

Since s ranges from 0 to 17 inclusive, the possible values of N are

$$51, 53, 55, \dots, 83, 85.$$

Searching through this list, we see that the primes are

$$53, 59, 61, 67, 71, 73, 79, 83,$$

for a total of $\boxed{8}$ primes.

Problem 1.3: Let $T = \text{TNYWR}$ and $f(x) = x^5 + 18x^4 + 19x^3 + 20x^2 + 21x + T$. The roots of f are a, b, c, d and e. Find $(a-1)(b-1)(c-1)(d-1)(e-1)$.

Solution. Notice that $f(x) = (x-a)(x-b)(x-c)(x-d)(x-e)$. Hence, we have $(a-1)(b-1)(c-1)(d-1)(e-1) = -f(1) = \boxed{-87}$.

Problem 2

Problem 2.1: $x, y \in \mathbb{R}$ satisfies $x\sqrt{y-1} + y\sqrt{x-1} = xy$. Find x.

Solution. From the problem, we know that $x, y > 1$. Hence, we suppose $x = \sec^2\alpha, y = \csc^2\beta$, where $0 < \alpha, \beta < \frac{\pi}{2}$. This gives

$$\sec^2\alpha\sqrt{\csc^2\beta - 1} + \csc^2\beta\sqrt{\sec^2\alpha - 1} = \sec^2\alpha\csc^2\beta,$$

which simplifies to $\sin 2\alpha + \sin 2\beta = 2$. Hence, we have $\sin 2\alpha = \sin 2\beta = 1$, which gives $\alpha = \beta = \frac{\pi}{4}$. Thus, we have $\boxed{x = 2}$.

Problem 2.2: Let $T = \text{TNYWR}$. A sequence $\{a_n\}$ satisfies that for any $m, n \in \mathbb{N}$ such that $m \geq n$ we have $a_{m+n} + a_{m-n} = \frac{1}{T}(a_{2m} + a_{2n})$. Given $a_1 = 1$, find the last digit of a_{2023}.

Solution. We observe that

$$\frac{1}{2}(a_{2m} + a_{2m}) = a_{2m} + a_0 = 2(a_m + a_m),$$

which gives $a_0 = 0$ and $a_{2m} = 4a_m$. Then, we can easily show that $a_m = m^2$ with induction. Hence, the last digit of a_{2023} is $\boxed{9}$.

Problem 2.3: Let $T = \text{TNYWR}$. The sequence $\{a_n\}$ satisfies $a_1 = 7$ and the recurrence relation

$$a_{n+1} = Ta_n + 7.$$

Find the sum of all values of i such that a_i is a divisor of a_{88}.

Solution. We write the numbers in base 9. Then note that

$$a_1 = (7)_9, \quad a_2 = (77)_9, \quad a_3 = (777)_9, \quad \cdots$$

This pattern holds because multiplying a_n by 9 moves the decimal place over when in base 9. So, in general, a_n is n sevens in base 9. Now we consider when $a_i \mid a_m$ for positive integers $i \leq m$. To do so, note that if we use long division, we get that

$$a_m \equiv a_{m \pmod i} \pmod{a_i},$$

where we let $a_0 = 0$ for notational purposes. From this, it's clear that $a_i \mid a_m$ iff $i | m$. So, our answer is the sum of the divisors of $88 = 2^3 \cdot 11$, which is

$$(1 + 2 + 2^2 + 2^3)(1 + 11) = (15)(12) = \boxed{180}.$$