Tiebreaker Round Solutions

DMM 2022

1 Tiebreaker

Problem 1: The sequence $\{x_n\}$ is defined by

$$x_{n+1} = \begin{cases} 2x_n - 1, & \text{if } \frac{1}{2} \leq x_n < 1 \\ 2x_n, & \text{if } 0 \leq x_n < \frac{1}{2} \end{cases}$$

where $0 \leq x_0 < 1$ and $x_7 = x_0$. Find the number of sequences satisfying these conditions.

Solution. First, we observe that the sequence $\{x_n\}$ is fully determined once x_0 is determined. Hence, we essentially need to find the number of different choices for x_0.

We represent the numbers in binary. Suppose $x_n = (0.b_1b_2\cdots)_2$. If $b_1 = 1$, then we have $\frac{1}{2} \leq x_n < 1$ and $x_{n+1} = 2x_n - 1 = (0.b_2b_3\cdots)_2$. If $b_1 = 0$, then we have $0 \leq x_n < \frac{1}{2}$ and $x_{n+1} = 2x_n = (0.b_2b_3\cdots)_2$. Hence, given $x_n = (0.b_1b_2\cdots)_2$, we always have $x_{n+1} = (0.b_2b_3\cdots)_2$.

Suppose $x_0 = (0.a_1a_2\cdots)_2$. Then, we have $x_7 = (0.a_8a_9\cdots)_2$. Since $x_0 = x_7$, we have $a_i = a_{i+7}$ for all $i \in \mathbb{N}^*$. Hence, we only need to count the number of choices for $\{a_1, \ldots, a_7\}$. Since we have 2 choices for each of them, we have $2^7 - 1 = \boxed{127}$ choices in total.

Problem 2: Let $M = \{1, \ldots, 2022\}$. For any nonempty set $X \subseteq M$, let a_X be the sum of the maximum and the minimum number of X. Find the average value of a_X across all nonempty subsets X of M.

Solution. For any nonempty subset X of M, let $X' = \{2023 - x \mid x \in X\}$. Then, X' is a nonempty subset of M as well. Moreover, if $X \neq Y$, then $X' \neq Y'$. Hence, we can pair up X and X', so that $a_X + a_{X'} = 2 \cdot 2023 = 4046$ by definition of X'. The only case in which we can not do such pairing is when $X = X'$. However, we must have $a_X = 2023$ in this case. Hence, the average value of a_X across all nonempty subsets X of M is $\boxed{2023}$.