2023 DUKE MATH MEET INDIVIDUAL ROUND

Problems 1-2 **Name** ________________

Time Limit: 10 minutes **Team** ________________

Problem 1 Let f be a function such that $f(x+y) = f(x) + f(y) - 1$ for all reals x, y. If $f(3) = 2$, find $f(15)$.

Problem 2 How many ways are there to arrange the integers 1 through 7 inclusive in a circle, where at most one pair of adjacent numbers in the circle have the same parity? Two numbers are said to have the same parity if they are either both even or both odd. (Rotations of an arrangement are considered to be the same arrangement).

ANSWER TO PROBLEM 1

ANSWER TO PROBLEM 2

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Problems 3-4 **Name** ________________

Time Limit: 10 minutes **Team** ________________

Problem 3 The sum of the 80 smallest positive solutions to the equation $\sin x = \cos 2x$ is $m\pi$, for some positive integer m. Find m.

Problem 4 The greater of the two real solutions of the following equation can be expressed as $\frac{a+\sqrt{b}}{c}$, where a, b, c are integers and b is not divisible by the square of any prime. Find $a + b + c$.

$$(3z + 1)(4z + 1)(6z + 1)(12z + 1) = 2$$

ANSWER TO PROBLEM 3

ANSWER TO PROBLEM 4

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Problems 5-6 **Name** ____________________

Time Limit: 10 minutes **Team** ____________________

Problem 5 Let n be the largest integer such that 17^n divides $\frac{(2023^2)!}{2023^{2023}}$. Write $n = 2023a + b$, where a and b are positive integers and $b < 2023$. Find $a + b$. (Note that $2023 = 7 \times 17^2$.)

Problem 6 Bob has a deck of 60 cards numbered from 1 to 60. Bob randomly distributes the cards into 30 pairs where each pair has exactly 2 cards. Then, Bob discards the smaller card within each pair and sums the remaining 30 cards. What is the expected value of the sum?

ANSWER TO PROBLEM 5

ANSWER TO PROBLEM 6

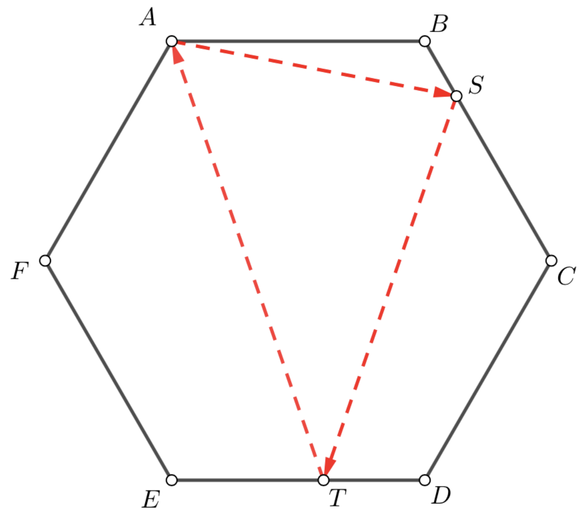
2023 DUKE MATH MEET INDIVIDUAL ROUND

Problems 7-8 **Name** ___________________

Time Limit: 10 minutes **Team** ___________________

Problem 7 There are six mirrors of the same length arranged into a regular hexagon, where the faces are put inwards. A beam of light passed through A, hits the mirrors and reflects for exactly 2 times, then goes back to A again. Denote S, T the first and the second time that the light hits the mirrors (as shown in the figure).

What is the ratio of $\frac{DT}{ET}$?



Problem 8 In a 3×3 grid, label each square with integers from 1 to 9 distinctly such that the number in each square is always smaller than both the numbers in the squares directly above it and directly to the left of it. Find the total number of all such possible labelings.

ANSWER TO PROBLEM 7

ANSWER TO PROBLEM 8

2023 DUKE MATH MEET INDIVIDUAL ROUND

Problems 9-10 **Name** __________________

Time Limit: 10 minutes **Team** __________________

Problem 9 A *perfect number* is a number that is equal to the sum of its divisors not including itself. The first three *perfect numbers* are 6, 28, and 496. Find the sum of the reciprocals of each divisor (including itself) of the fourth *perfect number*.

Problem 10 Define $f_n(k)$ to be the number of 0s in the n-digit binary form of k. For example, $f_5(5) = 3$ since $5 = 00101_2$ is five digits long and has three 0s. Let

$$S = \sum_{k=0}^{2^5-1} (-1)^{f_5(k)} 2^k$$

What are the last two digits of S?

ANSWER TO PROBLEM 9

ANSWER TO PROBLEM 10