Individual Round Solutions

DMM 2023

1 Individual Round

1. Let f be a function such that $f(x+y) = f(x) + f(y) - 1$ for all reals x, y. If $f(3) = 2$, find $f(15)$.

   Solution. The answer is $\boxed{6}$. Let $g(x) = f(x) - 1$. Then $g(x+y) = f(x+y) - 1 = f(x) - 1 + f(y) - 1 = g(x) + g(y)$. Since $g(3) = f(3) - 1 = 1$, we have $g(15) = g(3) + g(3) + g(3) + g(3) + g(3) = 5$, so $f(15) = g(15) + 1 = 6$. Note that $f(x) = x/3 + 1$ satisfies all criteria in the problem.

2. How many ways are there to arrange the integers 1 through 7 inclusive in a circle, where at most one pair of adjacent numbers in the circle have the same parity? Two numbers are said to have the same parity if they are either both even or both odd. (Rotations of an arrangement are considered to be the same arrangement).

   Solution. We see that there are 4 odd numbers and 3 even numbers. Let's replace the odd numbers with 1s and the even numbers with 0s and see how many valid arrangements there are. In this version, there is exactly one position in the arrangement where adjacent elements are both 1. All other elements of the circle alternate between 0 and 1.

   We see that after choosing the position of the adjacent elements with equal value, the rest of the arrangement is fixed. Thus there are 7 valid arrangements - one for each possible position of the equal adjacent elements. Now, to replace the 1s and 0s with the numbers from 1 to 7, there are 4! ways to replace the four 1s and 3! ways to replace the there 0s. Finally, we divide by 7 because we are counting each rotation of an arrangement 7 times. Thus the final answer is $7 \cdot 4! \cdot 3!/7 = \boxed{144}$.

3. The sum of the 80 smallest positive solutions to the equation $\sin x = \cos 2x$ is $m\pi$, for some positive integer m. Find m.

   Solution. The answer is $\boxed{2120}$. Substitute $\cos 2x = 1 - 2\sin^2 x$, we have $2\sin^2 x + \sin x - 1 = 0 \Leftrightarrow (2\sin x - 1)(\sin x + 1) = 0$. Therefore $\sin x = \frac{1}{2}, -1$. Therefore positive solutions are all in the form $x = \frac{\pi}{6} + 2k\pi$, $\frac{5\pi}{6} + 2k\pi$, or $\frac{3\pi}{2} + 2k\pi$, where k is a non-negative integer. Therefore the sum of the 80 smallest positive solutions is

$$\sum_{k=0}^{26}\left(\frac{\pi}{6} + 2k\pi\right) + \sum_{k=0}^{26}\left(\frac{5\pi}{6} + 2k\pi\right) + \sum_{k=0}^{25}\left(\frac{3\pi}{2} + 2k\pi\right) = 2120\pi$$

4. The greater of the two real solutions of the following equation can be expressed as $\frac{a+\sqrt{b}}{c}$, where a, b, c are integers and b is not divisible by the square of any prime. Find $a+b+c$.

$$(3z+1)(4z+1)(6z+1)(12z+1) = 2$$

Solution. We first multiply both sides by 8, 6, 4, and 2 to normalize the coefficient of z within each term to 24. This gives

$$(24z+8)(24z+6)(24z+4)(24z+2) = 16 \cdot 48.$$

We observe that the roots of the four terms are symmetric about $24z+5$, so we can apply difference of squares as follows:

$$\begin{aligned}
&(24z+8)(24z+6)(24z+4)(24z+2) \\
&= ((24z+5)+3)((24z+5)+1)((24z+5)-1)((24z+5)-3) \\
&= ((24z+5)^2-9)((24z+5)^2-1) \\
&= (((24z+5)^2-5)-4)(((24z+5)^2-5)+4) \\
&= ((24z+5)^2-5)^2-16
\end{aligned}$$

Setting this equal to $16 \cdot 48$ and solving yields $z = \frac{-5+\sqrt{33}}{24}$. Thus, the answer is $-5+33+24 = \boxed{52}$.

5. Let n be the largest integer such that 17^n divides $\frac{(2023^2)!}{2023^{2023}}$. Write $n = 2023a+b$, where a and b are positive integers and $b < 2023$. Find $a+b$. (Note that $2023 = 7 \times 17^2$.)

Solution. The answer is $\boxed{1008}$. Out of numbers from 1 to 2023^2, there exists $\frac{2023^2}{17}$ multiples of 17, $\frac{2023^2}{17^2}$ multiples of 17^2, $\frac{2023^2}{17^3}$ multiples of 17^3, and $\frac{2023^2}{17^4}$ multiples of 17^4. Since $2023^2 = 7^2 \times 17^4$, there are $\lfloor \frac{2023^2}{17^5} \rfloor = 2$ multiples of 17^5.

On the other hand, $2023^{2023} = 7^{2023} \times 17^{2\times 2023}$. Therefore $n = 2023\times 7\times 17 + 2023\times 7 + 7\times 7\times 17 + 7\times 7 + 2 - 2023\times 2$. Therefore $a = 7\times 17 + 7 - 2 = 124$, $b = 7\times 7\times 17 + 7\times 7 + 2 = 884$, and $a+b = 1008$.

6. Bob has a deck of 60 cards numbered from 1 to 60. Bob randomly distributes the cards into 30 pairs where each pair has exactly 2 cards. Then, Bob discards the smaller card within each pair and sums the remaining 30 cards. What is the expected value of the sum?

Solution. Intuitively, we will calculate the expectation by considering the probability that each card is present in the final sum.

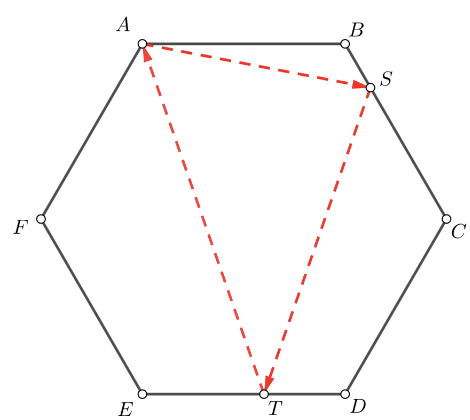
Let X be a random variable that represents the sum, and X_i be a random variable that represents the contribution of card i to the sum X, i.e., $X_i = i$ if it appears in the sum, and 0 otherwise. We see that $\mathbf{E}[X_i] = i \cdot P_i$, where P_i is the probability that card i appears in the sum. Card i appears in the sum if and only if it's matched with a smaller card. We can imagine first choosing card i to be part of a pair. Then out of the remaining 59 cards, $i-1$ of them are smaller than card i. Thus we have that $P_i = (i-1)/59$ and $\mathbf{E}[X_i] = i(i-1)/59$.

Applying linearity of expectation, we have

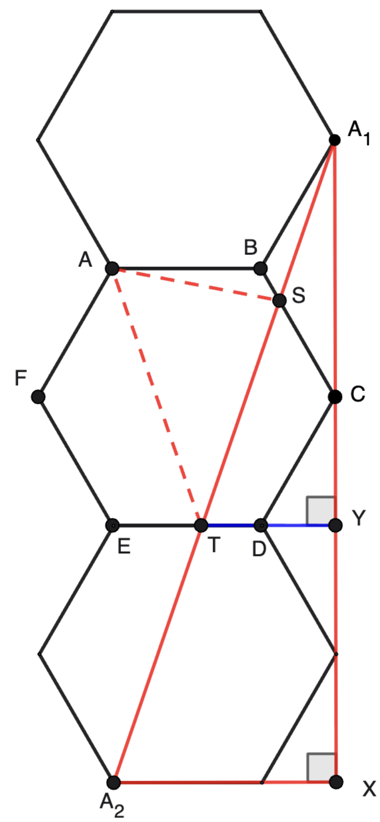
$$\begin{aligned}
\mathbf{E}[X] &= \mathbf{E}\left[\sum_{i=1}^{60} X_i\right] \\
&= \sum_{i=1}^{60} \mathbf{E}[X_i] \\
&= \sum_{i=1}^{60} i\frac{i-1}{59} \\
&= \frac{1}{59}\left(\sum_{i=1}^{60} i^2 - \sum_{i=1}^{60} i\right) \\
&= \frac{1}{59}\left(\frac{60(61)(121)}{6} - \frac{60(61)}{2}\right) \\
&= \frac{60(61)}{3} \\
&= \boxed{1220}
\end{aligned}$$

7. There are six mirrors of the same length arranged into a regular hexagon, where the faces are put inwards. A beam of light passed through A, hits the mirrors and reflects for exactly 2 times, then goes back to A again. Denote S, T the first and the second time that the light hits the mirrors (as shown in the figure).

   What is the ratio of $\frac{DT}{ET}$?



Solution. Let A_1 be the reflection of A over line BC, and A_2 be the reflection of A over line DE. Then, the points A_1, S, T, A_2 are collinear, as shown in the construction below:



By dropping a perpendicular from A_1, we form the right triangle $\triangle A_1XA_2$. Let Y be the intersection of $\overleftrightarrow{DE}$ and $\overline{A_1X}$. Then, we have $\triangle A_1YT \sim \triangle A_1XA_2$.

Using properties of 30-60-90 right triangles, we obtain that $A_2X = \frac{3}{2}DE$. Furthermore, notice that $A_1Y = \frac{3}{5}A_1X$ (informally, A_1 is 2.5 hexagons "tall" and Y is 1 hexagon "tall"). Thus, $TY = \frac{3}{5}A_2X = \frac{9}{10}DE$.

Note that $DY = \frac{1}{2}DE \implies DT = \frac{2}{5}DE \implies ET = DE - DT = \frac{3}{5}DE$. Thus, our answer is $\boxed{\frac{2}{3}}$.

8. In a 3×3 grid, label each square with integers from 1 to 9 distinctly such that the number in each square is always smaller than both the numbers in the squares directly above it and directly to the left of it. Find the total number of all such possible labelings.

Solution. The answer is $\boxed{42}$. For each square a_{ij} on the i-th row and j-th column, consider the set of squares

$$S_{ij} = \{a_{i'j'} \mid \text{ either } i = i', j \leq j' \text{ or } i \leq i', j = j'\}$$

Observe that $|S_{ij}| = i + j - 1$. For a fixed set of $|S_{ij}|$ distinct integers, there exists $|S_{ij}|!$ arbitrary different labelings on S_{ij}. However, since all elements in S_{ij} other than a_{ij} is either above or on the left of a_{ij}, a_{ij} must be labeled the smallest integer in S_{ij}. Therefore only $(|S_{ij}|-1)!$ different labelings, or $\frac{(|S_{ij}|-1)!}{|S_{ij}|!} = \frac{1}{|S_{ij}|}$ of all possible labelings would be valid.

Globally, there exists 9! arbitrary different labelings in total. To calculate the number of valid labelings, we compute

$$9! \prod_{a_{i,j}} \frac{1}{|S_{ij}|} = 9! \prod_{a_{i,j}} \frac{1}{i+j-1} = \frac{9!}{1 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 4 \cdot 3 \cdot 4 \cdot 5} = 42$$

9. A *perfect number* is a number that is equal to the sum of its divisors not including itself. The first three *perfect numbers* are 6, 28, and 496. Find the sum of the reciprocals of each divisor (including itself) of the fourth *perfect number*.

Solution. The answer is $\boxed{2}$. For a perfect number a, the answer is $\frac{1}{1} + \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n} + \frac{1}{a}$ where the k_is represent the divisors of a (excluding 1 and a itself). We can convert all of these to the denominator of a. Notice that $k_i k_{n-i} = a$ for all i. So the required sum is

$$\frac{a}{a} + \frac{k_n}{a} + \frac{k_{n-1}}{a} + \cdots + \frac{k_1}{a} + \frac{1}{a} = \frac{a + k_n + k_{n-1} + \ldots + 1}{a} = \frac{a+a}{a} = 2$$

since the sum of the divisors excluding a is equal to a. Notice it doesn't matter which perfect number we choose, they all have this property.

10. Define $f_n(k)$ to be the number of 0s in the n-digit binary form of k. For example, $f_5(5) = 3$ since $5 = 00101_2$ is five digits long and has three 0s. Let

$$S = \sum_{k=0}^{2^5-1} (-1)^{f_5(k)} 2^k$$

What are the last two digits of S?

Solution. Note that this sum can be expressed as the product

$$(2^{2^0} - 1)(2^{2^1} - 1)(2^{2^2} - 1)(2^{2^3} - 1)(2^{2^4} - 1)$$

A given term $(-1)^{f_5(k)} 2^k$ is generated by choosing 2^{2^m} if k has a 1 in the mth-place in binary, and -1 if there is a 0.

This product can also be derived recursively. Let

$$S_n = \sum_{k=0}^{2^n-1} (-1)^{f_n(k)} 2^k$$

To relate this to S_{n-1}, note that every n-digit binary string can be written as 1 or 0 appended to the front of an $n-1$-digit binary string. If 0 is appended, then this sum is equivalent to $-1 \cdot S_{n-1}$, since 2^k remains the same, so $f_5(k)$ increases by 1 so each term's sign is flipped. If

1 is appended, then this sum is equivalent to $2^{2^{n-1}} \cdot S_{n-1}$, since $f_5(k)$ remains the same, but we multiply by $2^{2^{n-1}}$ to append a 1 at the n-th place in front of each 2^k term. This yields

$$S_n = (2^{2^{n-1}} - 1)S_{n-1}$$

which yields the initial product.

To evaluate this modulo 100, note that the last we have

$$(2^{2^0} - 1)(2^{2^1} - 1)(2^{2^2} - 1)(2^{2^3} - 1)(2^{2^4} - 1) \equiv 1 \cdot 3 \cdot 15 \cdot 55 \cdot 35 \equiv \boxed{25} \pmod{100}$$