Power Round

DMM 2023

Magicians are full of tricks, but perhaps the best way to trick people is to use nontrivial math. In this power round, you will learn how to apply bits and graphs to perform one of the most famous math tricks - the *de Bruijn sequence*.

There are 70 points total. Additionally, in the "Directed Graphs" section there is an **open-ended bonus question**. This question **is not worth points**, but in rare situations may be used to determine team tie-breaks. **We strongly suggest you only work on this bonus problem if you are done with everything else.**

For questions asking you to **find**, **draw**, or **give**, you do not need to give any additional justification. There are no partial credits available for wrong solutions. For questions asking you to **show** or to **Explain with justification**, in order to receive full credits you should show a concrete, precise proof. Partial credits are available for these questions.

1 Introduction (13 pts)

The bit sequence 11000101 has an interesting feature: if you consider it as a cycle sequence, each possible binary word of length 3 is a contiguous subsequence in the sequence exactly once:

000 : 11**000**101
001 : 110**001**10
010 : 1100**010**1
011 : **1**10001**01**
100 : 1**100**0101
101 : 11000**101**
110 : **110**00101
111 : **11**00010**1**.

Such a sequence is called a *de Bruijn sequence*, or a *universal cycle* for the set of binary words of length 3. Sometimes, you can determine how many *universal cycles* are there, but in some cases, it remains an open question.

Problem 1:

(a) (2 pts) Find **one** universal cycle for the set of binary words of length 2.

(b) (3 pts) Find **one** universal cycle for the set of binary words of length 4.

Formally, we extend our definition of a de Bruijn sequence (or universal sequence) to a non-binary alphabet.

Definition: (de Bruijn sequence) *A de Bruijn sequence of rank n in an alphabet of k letters is a cyclic sequence of letters of length k^n, such that every sequence of letters of length n occurs precisely once as a contiguous subsequence.*

For example, if $n = 2, k = 3$, the following is an example of a rank 2 de Bruijn sequence with 3 letters.

$$\mathcal{F} = 012211020,$$

since for any two-letter word that has letters in $\{0, 1, 2\}$, the word is a contiguous subsequence of $\mathcal{F}$ exactly once.

Problem 2: (3 pts) Find **one** de Bruijn sequence with rank $n = 2$ and number of letters $k = 4$.

The de Bruijn sequence has many other variations, where we can consider cyclic sequence $\mathcal{F}$ of length s so that some sequences of letters (not necessarily all of them, in contrast to the definition of the normal de Bruijn sequence) appear exactly once in $\mathcal{F}$. Note that the existence of such a sequence can be no or an open-ended question.

Problem 3: We want to put $\binom{50}{5}$ numbers into a circle such that every set of five distinct integers from $\{1, 2, \ldots, 50\}$ appears somewhere on the circle at five consecutive positions (the order of the five in these five positions doesn't matter). Explain with justification these following questions

(a) (2 pts) Let x be an arbitrary integer in $\{1, 2, \ldots, 50\}$. Assume that there exists a way to put numbers as described above, how many contiguous $5-$tuples contain x?

(b) (3 pts) Show that there does not exist a way to put numbers that satisfy the above conditions.

2 Directed graph (39 pts)

In this section, our goal is to show that for any rank n and number of letters k, a de Bruijn sequence exists. To do so, we'll first equip with some concepts from graph theory.

To motivate our intuition into graph theory, let's consider the example of the de Bruijn sequence with $n = 2, k = 3$, as shown above, $\mathcal{F} = 012211020$. One can think about this sequence we start with 0, then we take a step to 1, then 2, and so on. Each step we take represents a sequence of 2 letters.

We can then represent our sequence into a graph of 3 vertices, where between two vertices, we draw a **directed** edge, and we also draw a directed edge from a vertex to itself. For example, with $n = 2, k = 3$, we have the graph in figure [1].

We can see that, our sequence $\mathcal{F}$ is an arrangement of all edges so that it goes in a complete, close cycle. Thus, finding one de Bruijn sequence is similar to finding a cycle in the constructed graph.

We formalize these notions by definitions

Definition: (Directed graph) *A directed graph G is a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and edges E, each edge map from a vertex v_i to another vertex v_j. Edges can map a vertex v_i to itself (this is called a self-loop), and we might have multiple edges between v_i to v_j in both directions.*

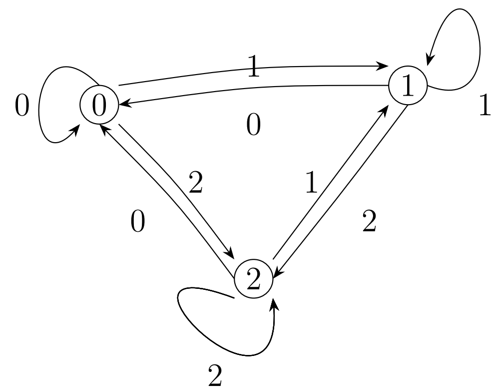

Figure 1: Example of a graph constructed from $\mathcal{F} = 012211020$.

Definition: (Strongly Connectedness) *We say that a directed graph $G = (V, E)$ is strongly connected if for any two vertices $x \neq y \in V$, there is a sequence of edges $x = v_1 \to v_2, v_2 \to v_3, \ldots, v_{k-1} \to v_k = y$ in E.*

Definition: (Weakly Connectedness) *We say that a directed graph $G = (V, E)$ is weakly connected, if we disregard the directions of all edges, then for any two vertices $x \neq y \in V$, there is a sequence of undirected edges from x to y.*

In short, a graph is strongly connected if one can get from any vertex to any other vertex by going along the edges (respecting the directions). A graph is weakly connected if one can get from any vertex to any other vertex by going along the edges, without considering their directions.

We can see that for a directed graph $G = (V, E)$, any vertex $v \in V$ has some edges that go into it, and some edges that go out of it. For a vertex v, denote the *indegree* of v as the number of edges $x \to v$ in E, and denote the outdegree of v as the number of edges $v \to x$ in E.

Definition: (Balance) *We say a directed graph G is balance if for each vertex, the in-degree is equal to the outdegree.*

Problem 4: (2 pts) Draw a strongly connected, balance directed graph with 5 vertices and 10 edges.

Problem 5: (2 pts) What is the smallest number of edges of a strongly connected, balance directed graph G with n vertices? (justify with full proof)

In a graph, we define a *cycle*, which is a set of edges $v_1 \to v_2 \to \ldots \to v_k \to v_1$. We are interested in determining a sorted order of edges so that it can form loop. This is called an *Eulerian path*.

Definition: (Eulerian path) *A Eulerian path in $G = (V, E)$ is a closed walk along edges of the graph which uses each edge exactly once.*

Problem 6: (2 pts) With the graph you constructed in problem 4, find an Eulerian path or argue that it does not exist. (Make sure that your solution to problem 4 is correct to receive full credits.)

For the following problem, we always assume that G has no isolated vertices, i.e., vertices that have no incoming or emanating edges.

Problem 7: Let G be a directed graph with no isolated vertices,

(a) (2 pts) Show that if G has an Eulerian path, then G is strongly connected and balance.

(b) (2 pts) Assume that G is balance and strongly connected, show that we can find a cycle C in G, and removing C from G will still make the graph balance.

(c) (3 pts) Show that if a directed graph is balance and weakly connected, then it is strongly connected.

(d) (3 pts) Show that if a directed graph is strongly connected and balance, then it has an Eulerian path.

We have determined the necessary and sufficient conditions for a directed graph to have an Eulerian path. We can now show that for any rank n and number of letters k, a de Bruijn sequence exists.

Given n, k, construct a graph $G = (V, E)$ as follows: V has k^{n-1} vertices, each corresponds to a unique sequence of $n - 1$ letters from the alphabet of k letters. Between two vertices A, B, there is an edge $A \to B$ if and only if $A = a_1 a_2 \ldots a_{n-1}$, and $B = a_2 \ldots a_{n-1} h$ for some letter h.

Problem 8:

(a) (2 pts) Draw the graph for $n = 4, k = 2$.

(b) (3 pts) Show that the constructed graph is balance and strongly connected for any n, k.

(c) (3 pts) Show that the de Bruijn sequence of rank n with k letters always exists.

We can create an algorithm to find one de Bruijn sequence as well.

Problem 9:

a) (4 pts) Let $k = 2$. Prove that a de Bruijn sequence of 0's and 1's of rank n can be constructed via the following algorithm. Start with $n - 1$ consecutive 0's and start adding symbols via the following rule. At each step add 1 if it doesn't cause repeating subsequences of length n, otherwise, add 0. Do $2^n - n + 1$ steps and consider the result as a cyclic sequence.

b) (6 pts) Modify (with proof) the above algorithm to construct a de Bruijn sequence for any n, k.

Once you grasp the essential ideas of the proofs, answer the following extension.

Problem 10: (5 pts) For which n can one put $\binom{n}{2}$ integers on a circle so that every two distinct integers from $\{1, 2, \ldots, n\}$ occur somewhere on a circle as two consecutive integers? (justify with full proof)

The following is an open-ended question, but partial progress can be awarded with partial credits.

Open Ended Question. Refer to instructions: For which values of n, k can one put $\binom{n}{k}$ integers on a circle so that every two distinct integers from $\{1, 2, \ldots, n\}$ occur somewhere on the circle as k consecutive integers? Partial credit is available if you manage to prove some specific cases of n, k.

3 Magic tricks (18 pts)

Now, with the existence of a de Bruijn sequence (and a nice algorithm to determine it), we are ready to do some magic!

A magician has a deck of 32 different cards, which he sorted in some ways before doing the trick. He then passes the deck to a random audience and asks them to cut the deck a few times (cut the deck here means they split the deck into two halves and swapped the halves).

Then, the magicians will select 5 random people in the audience, and each of them will take the card from the top, one by one. The magician says that he has a special telepathy ability that he can divine the cards that each person is holding, yet his ability is somewhat restricted and he has to ask them some questions. He then proceeded to ask each person if their holding card was red or black, and then, he could guess the exact card of each person.

Problem 12: (3 pts) Explain, with justification, how this trick works. You can assume that the magician has a talent for memorizing exactly the ordering of the cards.

Problem 13: (3 pts) The audience challenges the magician to perform the same trick, but with a deck of 33 cards. Do you think the magician can create some tricks that would guess all the cards exactly? Explain with justification.

Now, a magician wants to modify his trick a bit. He selects from the deck K cards of his choice, and he arranges it in some ways. He again asks one random audience to cut the deck, then he asks 4 random audiences to sequentially pick the top card, one by one. He then asks the 4 people to group into different houses, i.e., if two people have two cards of club, they will go into the same group. Note that the magician does not know which group corresponds to which house. The magician sees this and then he can decide their cards exactly.

Problem 14: (8 pts) Explain with justification, what is the maximum number K that the magician can have a working trick? With that K, describe how he can perform the trick.

The magician can perform the same trick with 5 people selecting cards, with a deck of cards of exactly 52 cards! (using exactly 13 each of clubs, hearts, spades, and diamonds, then swap the card to Joker).

Problem 15: (4 pts) Show that 52 is the maximum amount of cards that the magician can perform the trick successfully, assuming that he selects 5 people instead of 4.

You won't be asked to derive a strategy with 52 cards since it is beyond the scope of 1 hour to answer this question. However, you can think more about how you can help the magician perform his trick since you might be able to use these tricks yourself to impress your friends!