# Power Round Solutions

DMM 2023

Magicians are full of tricks, but perhaps the best way to trick people is to use nontrivial math. In this power round, you will learn how to apply bits and graphs to perform one of the most famous math tricks - the *de Bruijn sequence*.

There are 70 points total. Additionally, in the "Directed Graphs" section there is an **open-ended bonus question**. This question **is not worth points**, but in rare situations may be used to determine team tie-breaks. **We strongly suggest you only work on this problem if you are done with everything else.**

## 1 Introduction (13 pts)

The bit sequence 11000101 has an interesting feature: if you consider it as a cycle sequence, each possible binary word of length 3 is a contiguous subsequence in the sequence exactly once:

000 : 11**000**101
001 : 110**001**10
010 : 1100**010**1
011 : **1**10001**01**
100 : 1**100**0101
101 : 11000**101**
110 : **110**00101
111 : **11**00010**1**.

Such a sequence is called a *de Bruijn sequence*, or a *universal cycle* for the set of binary words of length 3. Sometimes, you can determine how many *universal cycles* are there, but in some cases, it remains an open question.

**Problem 1:**

**For this problem, to check by definition if the students' solution is correct.**

(a) (2 pts) Find **one** universal cycle for the set of binary words of length 2.

1100. There can be many correct solutions.

(b) (3 pts) Find **one** universal cycle for the set of binary words of length 4.

0000100110101111. There can be many correct solutions.

Formally, we extend our definition of a de Bruijn sequence (or universal sequence) to a non-binary alphabet.

**Definition:** (de Bruijn sequence) *A de Bruijn sequence of rank $n$ in an alphabet of $k$ letters is a cyclic sequence of letters of length $k^n$, such that every sequence of letters of length $n$ occurs precisely once as a contiguous subsequence.*

For example, if $n = 2, k = 3$, the following is an example of a rank 2 de Bruijn sequence with 3 letters.

$$\mathcal{F} = 012211020,$$

since for any two-letter word that has letters in $\{0, 1, 2\}$, the word is a contiguous subsequence of $\mathcal{F}$ exactly once.

**Problem 2:** (3 pts) Find **one** de Bruijn sequence with rank $n = 2$ and number of letters $k = 4$.

One example is 0112233021320310.

The de Bruijn sequence has many other variations, where we can consider cyclic sequence $\mathcal{F}$ of length $s$ so that some sequences of letters (not necessarily all of them, in contrast to the definition of the normal de Bruijn sequence) appear exactly once in $\mathcal{F}$. Note that the existence of such a sequence can be no or an open-ended question.

**Problem 3:** We want to put $\binom{50}{5}$ numbers into a circle such that every set of five distinct integers from $\{1, 2, \ldots, 50\}$ appears somewhere on the circle at five consecutive positions (the order of the five in these five positions doesn't matter).

(a) (2 pts) Let $x$ be an arbitrary integer in $\{1, 2, \ldots, 50\}$. Assume that there exists a way to put numbers as described above, how many contiguous $5-$tuples contain $x$?

If there exists a way to put the numbers as described, then there is $\binom{99}{4}$ ways to pick 4 numbers from the remaining 49 numbers. Hence, $x$ appears in $\binom{49}{4}$ contiguous $5-$tuples.

(b) (3 pts) Show that there does not exist a way to put numbers that satisfy the above conditions.

If there exists such a way, then first note that any 5 consecutive numbers must be different. Since we have $\binom{50}{5}$ total consecutive $5-$tuples, and thus all different subset of 5 distinct integers from $\{1, 2, \ldots 50\}$ appears somewhere among these consecutive tuples. Now, for a number $x$, there are exactly 5 consecutive $5-$tuples containing $x$. Since none of those tuples contain two different numbers $x$, we must have the number of tuples containing $x$ divisible by 5. Note that 5 does not divide $\binom{49}{4}$, so we have a contradiction.

## 2 Directed graph (39 pts)

In this section, our goal is to show that for any rank $n$ and number of letters $k$, a de Bruijn sequence exists. To do so, we'll first equip with some concepts from graph theory.

To motivate our intuition into graph theory, let's consider the example of the de Bruijn sequence with $n = 2, k = 3$, as shown above, $\mathcal{F} = 012211020$. One can think about this sequence we start with 0, then we take a step to 1, then 2, and so on. Each step we take represents a sequence of 2 letters.

We can then represent our sequence into a graph of 3 vertices, where between two vertices, we draw a **directed** edge, and we also draw a directed edge from a vertex to itself. For example, with $n = 2, k = 3$, we have the graph in figure [1].

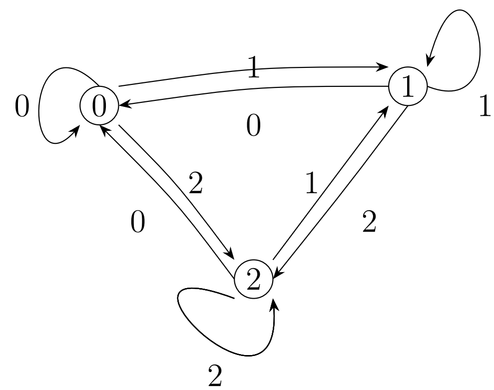


Figure 1: Example of a graph constructed from $\mathcal{F} = 012211020$.

We can see that, our sequence $\mathcal{F}$ is an arrangement of all edges so that it goes in a complete, close cycle. Thus, finding one de Bruijn sequence is similar to finding a cycle in the constructed graph.

We formalize these notions by definitions

**Definition:** (Directed graph) *A directed graph $G$ is a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$ and edges $E$, each edge map from a vertex $v_i$ to another vertex $v_j$. Edges can map a vertex $v_i$ to itself (this is called a self-loop), and we might have multiple edges between $v_i$ to $v_j$ in both directions.*

**Definition:** (Strongly Connectedness) *We say that a directed graph $G = (V, E)$ is strongly connected if for any two vertices $x \neq y \in V$, there is a sequence of edges $x = v_1 \to v_2, v_2 \to v_3, \ldots, v_{k-1} \to v_k = y$ in $E$.*

**Definition:** (Weakly Connectedness) *We say that a directed graph $G = (V, E)$ is weakly connected, if we disregard the directions of all edges, then for any two vertices $x \neq y \in V$, there is a sequence of undirected edges from $x$ to $y$.*

In short, a graph is strongly connected if one can get from any vertex to any other vertex by going along the edges (respecting the directions). A graph is weakly connected if one can get from any vertex to any other vertex by going along the edges, without considering their directions.

We can see that for a directed graph $G = (V, E)$, any vertex $v \in V$ has some edges that go into it, and some edges that go out of it. For a vertex $v$, denote the *indegree* of $v$ as the number of edges $x \to v$ in $E$, and denote the outdegree of $v$ as the number of edges $v \to x$ in $E$.

**Definition:** (Balance) *We say a directed graph $G$ is balance if for each vertex, the in-degree is equal to the outdegree.*

**Problem 4:** (2 pts) Draw a strongly connected, balance directed graph with 5 vertices and 10 edges.

An example of such graph is shown in the figure

**Problem 5:** (2 pts) What is the smallest number of edges of a strongly connected, balance directed graph $G$ with $n$ vertices?

The answer is $n$. For construction, take a loop through all $n$ vertices. For optimality, note that if the graph $G$ has at most $n-1$ edges, then there is one vertex with no out-degree (since the total sum of out degree is $n-1$), so from this vertex we can not go anywhere else.

In a graph, we define a *cycle*, which is a set of edges $v_1 \to v_2 \to \ldots \to v_k \to v_1$. We are interested in determining a sorted order of edges so that it can form loop. This is called an *Eulerian path*.

**Definition:** (Eulerian path) *A Eulerian path in $G = (V, E)$ is a closed walk along edges of the graph which uses each edge exactly once.*

**Problem 6:** (2 pts) With the graph you constructed in problem 4, find an Eulerian path or argue that it does not exist. (Make sure that your solution to problem 4 is correct to receive credits for this problem)

For the following problem, we always assume that $G$ has no isolated vertices, i.e., vertices that have no incoming or emanating edges.

**Problem 7:** Let $G$ be a directed graph with no isolated vertices,

(a) (2 pts) Show that if $G$ has an Eulerian path, then $G$ is strongly connected and balance.

Assume that an Eulerian tour exists. If we move along this tour, we come into each vertex as many times as we leave it, so the graph must be balanced. Next, while moving along the cycle one can get from any vertex to any other vertex, so the graph is strongly connected. (note that the assumption that the graph does not have any isolated vertices is important, because of this then the Eulerian path is guaranteed to reach any vertices.)

(b) (2 pts) Assume that $G$ is balance and strongly connected, show that we can find a cycle $C$ in $G$, and removing $C$ from $G$ will still make the graph balance.

Start from any vertex, then go along the edges until we hit some other vertices we already visited. We are guaranteed to hit some vertices again because otherwise, if we stuck at one vertex and fail to proceed, then the indegree to that vertex is larger than the out-degree from the vertex, contrary to the assumption that it is balance.

Note that in a cycle, the indegree and out-degree of any vertices are equal, so remove it will maintain the balanceness of the graph.

(c) (3 pts) Show that if a directed graph is balance and weakly connected, then it is strongly connected.

Take an arbitrary vertex $v$. Let $A$ be a set of vertices reachable from $v$ (in a strongly connected sense), and let $B$ be the set of all other vertices. Assume that $B$ is nonempty (this means that the graph is not strongly connected). Then there are no edges from $A$ to $B$, but there should be some edges from $B$ to $A$ due to the weak connectedness. Now comparing the total in- and out-degree of the vertices of $A$ we come to a contradiction: these sums should be equal due to balancedness, but the first one is greater due to the edges from $B$ to $A$, a contradiction. Indeed, let the total of in-degree of vertices in $A$ be decomposed into $d_A + d_B$, where $d_A$ are from vertices in $A$ and $d_B$ are from vertices in $B$, then $d_A$ is also the out-degree of all vertices in $A$, as from $A$ we can only go back to $A$. Since $d_B > 0$ this contradicts balanceness.

(d) (3 pts) Show that if a directed graph is strongly connected and balance, then it has an Eulerian path.

We proceed by induction. First of all we take a cycle $C$. Removing this cycle might disconnect the graph, and by (c) if the graph is strongly disconnected it must be weakly disconnected. Hence, the graph splits into several components each of which is balanced and connected, so each has an Eulerian tour (by induction). Then a desired tour can be constructed as follows: we move along the edges of $C$ until we come to some vertex of some component; then we walk along the Eulerian tour of this component, and then we move further along $C$, and so on.

We have determined the necessary and sufficient conditions for a directed graph to have an Eulerian path. We can now show that for any rank $n$ and number of letters $k$, a de Bruijn sequence exists.

Given $n, k$, construct a graph $G = (V, E)$ as follows: $V$ has $k^{n-1}$ vertices, each corresponds to a unique sequence of $n-1$ letters from the alphabet of $k$ letters. Between two vertices $A, B$, there is an edge $A \to B$ if and only if $A = a_1 a_2 \dots a_{n-1}$, and $B = a_2 \dots a_{n-1} h$ for some letter $h$.

**Problem 8:**

(a) (2 pts) Draw the graph for $n = 4, k = 2$.

Drawn during the grading session.

(b) (3 pts) Show that the constructed graph is balance and strongly connected for any $n, k$.

The in- and out-degree are $k$ so it is balance. Lastly, it is strongly connected since for every two vertices there exists a path connecting them (this path corresponds to a concatenation of these words).

(c) (3 pts) Show that the de Bruijn sequence of rank $n$ with $k$ letters always exists.

Take an Eulerian path in the above graph and traverse. We start building a sequence with the vertex $00\dots0$, then traverse along the edges, each time of traversal we attach the letter assigned to the edge. In the end, we obtain a sequence of length $n^k$. Each contiguous sequence of $n$ letters corresponds to $A + B$, where $A$ is a unique sequence of $n-1$ letters, and

$B$ is one of $k$ letters. This essentially generates all sequences of $n$ letters from the alphabet of $k$ elements.

We can create an algorithm to find one de Bruijn sequence as well.

**Problem 9:**

a) (4 pts) Let $k = 2$. Prove that a de Bruijn sequence of 0's and 1's of rank $n$ can be constructed via the following algorithm. Start with $n - 1$ consecutive 0's and start adding symbols via the following rule. At each step add 1 if it doesn't cause repeating subsequences of length $n$, otherwise, add 0. Do $2^n - n + 1$ steps and consider the result as a cyclic sequence.

b) (6 pts) Modify (with proof) the above algorithm to construct a de Bruijn sequence for any $n, k$.

A generalized algorithm would be as followed: starting from the sequence of $n - 1$ zeros, on each step put the maximal possible letter, such that no word of length $n$ is repeated. Proceed to do so for $k^n - n + 1$ times.

We record the first time this algorithm is stuck. Note that, at any point, for any vertex $x$ except for the vertex $00 \dots 0$, the number of inward edges to $x$ already visited is one count larger than number of ourward edges from $x$. Hence, if we can make a visit to $x$, we can exit $x$. So the only vertex that we will be stuck is $00 \dots 0$.

If we have unused edges, denote black edges the unused edges labelled with 0. We first show that there are no black edge cycle after we are stuck. Note that the edge from $00 \dots 0$ labelled 0 should be already visited. If there exists a black edge cycle, then this cycle will pass through $00 \dots 0$, which is a contradiction. Now, if there exists an unused edge between $v_1 \to v_2$, then there exists a black edge from $v_1 \to v_2$ (due to the fact that we always choose an edge label with the largest available integer). Note that since there are less than $k$ visited edges to $v_2$, there are less than $k$ visited edges out of $v_2$, and thus there is another black edge from $v_2 \to v_3$, and so forth. So we can form a black edge cycle, which is a contradiction.

Once you grasp the essential ideas of the proofs, answer the following extension.

**Problem 10:** (5 pts) For which $n$ can one put $\binom{n}{2}$ integers on a circle so that every two distinct integers from $\{1, 2, \dots, n\}$ occur somewhere on a circle as two consecutive integers?

The answer is $n$ odd. The corresponding graph will be the full graph $K_n$, and we only need to find an Eulerian path on this graph. Such a tour exists if $n$ is odd and not if $n$ is even.

The following is an open-ended question, but partial progress can be awarded with partial credits.

**Open Ended Question. Refer to instructions:** For which values of $n, k$ can one put $\binom{n}{k}$ integers on a circle so that every two distinct integers from $\{1, 2, \dots, n\}$ occur somewhere on the circle as $k$ consecutive integers? Partial credit is available if you manage to prove some specific cases of $n, k$.

A few ideas to extend to the generalize case:

1. From problem 3, we can see that if $k$ divides $n$ then such configuration does not exist.

2. It can be shown that if $n \geq 8$ and $n$ is not divisible by 3, then such configuration would exist.
3. (Open-ended) Does there exists a constant $A(k)$ such that if $n \geq A(k)$ and $n$ is not divisible by $k$, then there exists one configuration for such $n, k$?

# 3 Magic tricks (18 pts)

Now, with the existence of a de Bruijn sequence (and a nice algorithm to determine it), we are ready to do some magic!

A magician has a deck of 32 different cards, which he sorted in some ways before doing the trick. He then passes the deck to a random audience and asks them to cut the deck a few times (cut the deck here means they split the deck into two halves and swapped the halves).

Then, the magicians will select 5 random people in the audience, and each of them will take the card from the top, one by one. The magician says that he has a special telepathy ability that he can divine the cards that each person is holding, yet his ability is somewhat restricted and he has to ask them some questions. He then proceeded to ask each person if their holding card was red or black, and then, he could guess the exact card of each person.

**Problem 12:** (3 pts) Explain, with justification, how this trick works. You can assume that the magician has a talent for memorizing exactly the ordering of the cards.

Take a de Bruijn binary sequence of rank $n = 5$. Arrange the 32 cards in a way that red is 0, black is 1. Then, cutting the deck basically just rotating this sequence, which should not affect the fact that any $5-$consecutive sequences are unique. Five people taking the cards from the top will inform the magician about this order, and he can then guess correctly what is the card. Award +3, +2, +1, +0 depending on their argument.

**Problem 13:** (3 pts) The audience challenges the magician to perform the same trick, but with a deck of 33 cards. Do you think the magician can create some tricks that would guess all the cards exactly?

The answer is no. The magician can only infer from the five people selecting the cards and their colors. So his information is at most $2^5 = 32$. With only 32 different possible scenarios he can only distinguish up to 32 cards.

Now, a magician wants to modify his trick a bit. He selects from the deck $K$ cards of his choice, and he arranges it in some ways. He again asks one random audience to cut the deck, then he asks 4 random audiences to sequentially pick the top card, one by one. He then asks the 4 people to group into different houses, i.e., if two people have two cards of club, they will go into the same group. Note that the magician does not know which group corresponds to which house. The magician sees this and then he can decide their cards exactly.

**Problem 14:** (8 pts) What is the maximum number $K$ that the magician can have a working trick? With that $K$, describe how he can perform the trick.

In a similar fashion to problem 13, the maximum $K$ will be the number such that, among any four consecutive cards in the deck, the partition of the cards into four different subsets will be different. This $K$ is then the count for all possible partitions of $\{1, 2, 3, 4\}$ into 4 different, disjoint subsets. A simple counting gives $K = 15$. To show that this maximum is achieved, one example sequence is AAAABAABBCABDBD.

The magician can perform the same trick with 5 people selecting cards, with a deck of cards of exactly 52 cards! (using exactly 13 each of clubs, hearts, spades, and diamonds, then swap the card to Joker).

Problem 15 was initially intended to give only a bound. It is not intended to show that 52 is the maximum number.

**Problem 15:** (4 pts) Show that the magician can not perform the tricks with 53 cards, assuming that he selects 5 people instead of 4.

Similar to problem 14, the objective of this problem is to show that there are 52 different ways to partition $\{1, 2, 3, 4, 5\}$ into 4 different, disjoint subsets.

You won't be asked to derive a strategy with 52 cards since it is beyond the scope of 1 hour to answer this question. However, you can think more about how you can help the magician perform his trick since you might be able to use these tricks yourself to impress your friends!