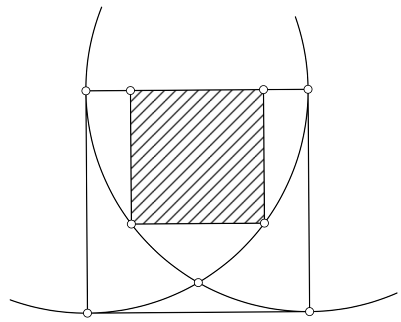
2023 DUKE MATH MEET TEAM ROUND

1. There is a square of side length 5, and there are two quarters of a circle centered at two adjacent vertices of the square with radii 5 (as shown in the figure). Another square is inscribed inside the original square and two quarters. What is the area of this square?

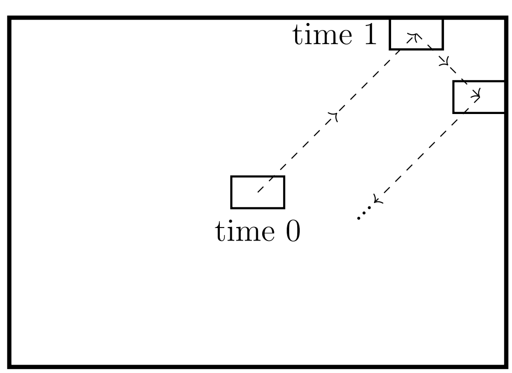


2. In 3-dimensional space, let $P = \{(x, y, z) \mid x, y, z \in \{0, 1, 2, 3\}\}$. Find the total number of lines that pass through at least 2 points in P.

3. A *strange* number is a positive integers n that satisfies $\left\lfloor \frac{n}{2} \right\rfloor \equiv 3 \pmod 4$, $\left\lfloor \frac{n}{8} \right\rfloor \equiv 2 \pmod 4$, $\left\lfloor \frac{n}{64} \right\rfloor \equiv 1 \pmod 4$. What is the 70^{th} smallest strange number?

4. A plane cuts a sphere of radius 6 so that the intersection is a circle. Three points A, B, C are chosen on this circle that form an equilateral triangle of side 6. The planes through A, B, and C that are tangent to the sphere intersect at a point D. Let the distance from D to the center of the sphere be d. Find d^2.

5. A TV screen of width 47 and height 33 has a DVD logo of width 5 and height 3 that begins centered on the screen. The DVD logo begins moving at time 0 along a line with slope 1, and hits an edge (its edge and the TV screen's edge coincide) for the first time at time 1. Whenever the logo hits an edge, it reflects off of it but continues travelling at the same speed. What is the time when the logo first hits a corner (*i.e.*, hits two edges simultaneously)?



6. In Duke there's a tradition of tenting before big games. At least a third of all people must be in the tent at all times. A group of 6 create a schedule where they split the day into 3

timeslots, and they'll assign people to a timeslot which they'll take each day. Those that take the night timeslot will also only be assigned 1 slot. They agree that everyone should take at least 1 timeslot. How many ways can they assign the timeslots to the 6 people?

7. Find the number of ordered triplets of integers (x, y, z) such that

$$(x+y)^2 + (y+z)^2 + (z+x)^2 + (x+y)(x+z) + (y+z)(y+x) + (z+x)(z+y) = 328.$$

8. Find the greatest positive constant λ such that $x^5 + \frac{1}{x^5} - 2 \geq \lambda(x + \frac{1}{x} - 2)$ for all $x > 0$.

9. Eric and Julia like to spend their nights factoring. Last night they factored 2023. It's just $7 \cdot 17^2$, but everyone knows that! Tonight, they have a trickier problem in store. What's the smallest divisor of $2023^4 + 9^4$ that is greater than 2?

10. In the variant *duck chess*, a rubber duck is a piece that can block a pair of rooks from attacking each other by acting as a barrier. On a 5×5 chessboard, or grid of squares, we can place exactly one piece on each square. Let a configuration of six rooks and two rubber ducks on the chessboard be *valid* if it satisfies the following:

    (a) With the six rooks and two rubber ducks placed, no pair of rooks attack each other

    (b) If the rubber ducks are removed, then there exists a rook such that if we remove it, there exists no pair of rooks that attack each other amongst the remaining five

    How many *valid* configurations are there? An example of one is shown below. (Note: A *rook* is a piece that is said to attack a square if it is on the same row or column and there exist no pieces between the rook and the tile)