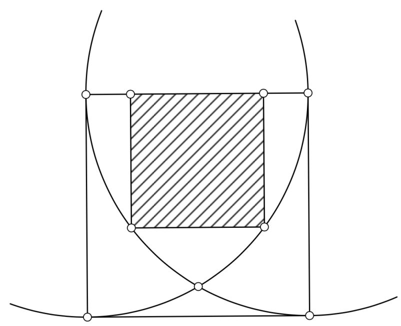
Team Round Solutions

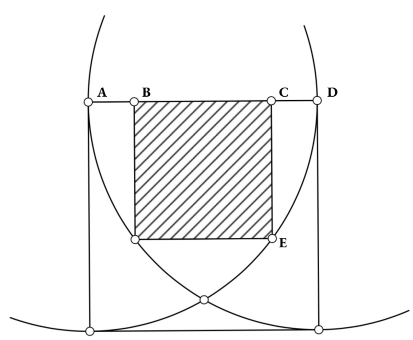
DMM 2023

1 Team Round

1. There is a square of side length 5, and there are two quarters of a circle centered at two adjacent vertices of the square with radii 5 (as shown in the figure). Another square is inscribed inside the original square and two quarters. What is the area of this square?



Solution. Label the points A, B, C, D, E as below:



Let $\overline{AB} = \overline{CD} = x$ and $\overline{BC} = s$. This gives us that $2x + s = 5$. Note that $\overline{AE} = 5$ as it a radius of the circle, so we have the following equation via the Pythagorean theorem: $s^2 + (s + x)^2 = 25$. Solving those two equations for x and s and taking the positive solution for s results in $s = 3$, which makes the area of the inner square $\boxed{9}$

2. In 3-dimensional space, let $P = \{(x, y, z) \mid x, y, z \in \{0, 1, 2, 3\}\}$. Find the total number of lines that pass through at least 2 points in P.

Solution. The answer is $\boxed{1492}$. Not regarding multiples, we can find $\binom{4^3}{2} = 2016$ lines by connecting every pair of points in P. Now we consider multiples.

There are $16 \times 3 + 4 \times 6 + 4 = 76$ lines that pass through 4 points. This consists of 16 grid lines (parallel to the sides of the cube) in each of the three dimensions, 4 diagonals (parallel to face diagonals) connecting each of the 6 pairs of diagonally opposite edges, and 4 body diagonals of the cube. Each of these lines is counted $\binom{4}{2} - 1 = 5$ times excessively.

3. A *strange* number is a positive integers n that satisfies $\lfloor \frac{n}{2} \rfloor \equiv 3 \pmod 4$, $\lfloor \frac{n}{8} \rfloor \equiv 2 \pmod 4$, $\lfloor \frac{n}{64} \rfloor \equiv 1 \pmod 4$. What is the 70^{th} smallest strange number?

   Solution. Consider the binary representation of a strange number n. Informally, the equality $\lfloor \frac{n}{2} \rfloor \equiv 3 \pmod 4$ means that, when the last digit of n is removed, the last two digits are 11. Thus, we have that the last three digits of n are of the form $11a$ where a is either 0 or 1. Similarly, the next two equalities imply that the last five digits of n are of the form $10abc$, and that the last eight digits of n are of the form $01abcdef$, where letters represent either 0 or 1.

   Combining these statements, we have that the last eight digits of n are of the form $01a1011b$. Note that there are 4 total choices of a and b combined. This means that, for the 70^{th} strange number n, $ab_2 \equiv 69 \equiv 01_2 \pmod 4$, and the ninth digit and beyond of n are the number $\lfloor \frac{69}{4} \rfloor = 17 = 10001_2$. Thus, the 70^{th} strange number is $n = 1000101010111_2 = \boxed{4439}$

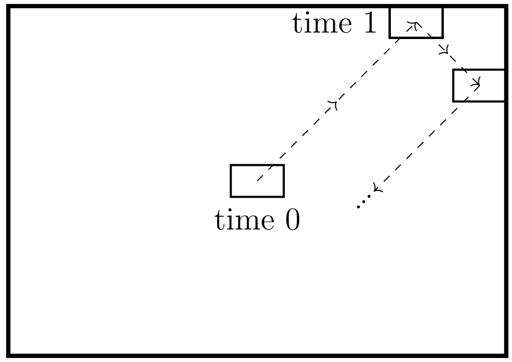
4. A plane cuts a sphere of radius 6 so that the intersection is a circle. Three points A, B, C are chosen on this circle that form an equilateral triangle of side 6. The planes through A, B, and C that are tangent to the sphere intersect at a point D. Let the distance from D to the center of the sphere be d. Find d^2.

   Solution. The answer is $\boxed{54}$. A, B, C and the center O of the sphere form a regular tetrahedron of side 6. AD, BD, and CD must be perpendicular to AO, BO, and CO respectively because D is on each of the planes that are tangent to the sphere. Now since $AO = 6$, we know that $AD = \sqrt{d^2 - 36}$ since ADO is a right triangle. Let M be the intersection of DO and the plane (ABC). Notice that DO is perpendicular to (ABC) through the center of the triangle ABC because of symmetry. This means that MO is the height of the tetrahedron, which we can calculate to be $2\sqrt{6}$. This means $DM = d - 2\sqrt{6}$. Now observe the triangle ADM, which is a right triangle as DM is perpendicular to (ABC). We can calculate $AM = 2\sqrt{3}$ since ABC is an equilateral triangle. We know $AD = \sqrt{d^2 - 36}$ from before, so by the Pythagorean theorem,

$$AM^2 + DM^2 = AD^2 \Rightarrow 12 + (d - 2\sqrt{6})^2 = d^2 - 36$$

   Solving this for d, we get $d = 3\sqrt{6}$. This gives us our answer as $d^2 = (3\sqrt{6})^2 = 54$.

5. A TV screen of width 47 and height 33 has a DVD logo of width 5 and height 3 that begins centered on the screen. The DVD logo begins moving at time 0 along a line with slope 1, and hits an edge (its edge and the TV screen's edge coincide) for the first time at time 1. Whenever the logo hits an edge, it reflects off of it but continues travelling at the same speed. What is the time when the logo first hits a corner (*i.e.*, hits two edges simultaneously)?



Solution. The answer is $\boxed{7}$. The center of the DVD logo is restricted to a 42×30 region and starts at $(21, 15)$. Notice that reflecting off a wall can be interpreted as going into another screen that is reflected along the shared border. Then, DVD logo hits a corner when the position of its center is (x, y) such that $42 \mid x$ and $30 \mid y$. Since the logo travels along a line with slope 1, its position can always be described by $(21 + z, 15 + z)$, so it suffices to find the smallest positive integer z such that $21 + z \equiv 0 \pmod{42}, 15 + z \equiv 0 \pmod{30}$. Thus, $z = 42n - 21 = 30m - 15$ for nonnegative integers n, m. Simplifying, $n = \frac{30m+6}{42} = \frac{5m+1}{7}$. Thus, the least n, m for which both are nonnegative integers is $n = 3, m = 4$, so $z = 105$. Thus, the total distance travelled by the DVD logo is $105\sqrt{2}$. Additionally, it reaches the first edge after travelling a distance of $15\sqrt{2}$ (since the top edge is closer) at a time 1, so the total time taken is $\frac{105\sqrt{2}}{15\sqrt{2}} = 7$.

6. In Duke there's a tradition of tenting before big games. At least a third of all people must be in the tent at all times. A group of 6 create a schedule where they split the day into 3 timeslots, and they'll assign people to a timeslot which they'll take each day. Those that take the night timeslot will also only be assigned 1 slot. They agree that everyone should take at least 1 timeslot. How many ways can they assign the timeslots to the 6 people?

Solution. The answer is $\boxed{1220}$. Observe the night timeslot. We need at least 2 people in this slot, but can't have more than 4 since we need at least 2 people to take the other two slots. Let's do some casework on the number of people taking the night slot:

- 2 people taking night timeslot: there are $\binom{6}{2} = 15$ ways to choose these 2 people. This means there are 4 people taking the other two slots together.
  - If 4 people are taking the first slot, we need either 2, 3, or 4 people for the second slot. This contributes $\binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 11$ ways.
  - If 3 people are taking the first slot, then we need the fourth person to take the second one, and we need one more so all subsets of the first 3 people work except the empty one. There are 4 ways to choose the first 3 people for the first slot, and $2^3 - 1 = 7$ ways to choose people for the second slot, so this contributes 28 ways.
  - If only 2 people are taking the first slot, the other two must be on the second slot, and any subset of the first two can join them in the second slot. This contributes

$\binom{4}{2} \times 4 = 24$ ways.

This gives us $15 \times (11 + 28 + 24) = 945$ ways for this case.

- 3 people taking night timeslot: there are $\binom{6}{3} = 20$ ways to choose these 3 people. This means there are 3 people taking the other two slots together.
  - If all 3 people take the first slot, we can have any 2 or 3 people take the second. This contributes $\binom{3}{2} + \binom{4}{3} = 4$ ways.
  - If 2 people are taking the first slot, the last person must take the second slot. Then we need 1 or 2 people from the first slot to also take the second slot. There are 3 ways to choose 2 people for the first slot, and another $2^2 - 1 = 3$ ways to choose people for the second slot. This contributes 9 ways.

  This gives us $20 \times (4 + 9) = 260$ ways for this case.

- 4 people taking night timeslot: there are $\binom{6}{4} = 15$ ways to choose these 4 people. This means there are 2 people taking the other two slots together. This one is simple, both people have to take both slots so this contributes $15 \times 1 = 15$ ways.

In total, we have $945 + 260 + 15 = 1220$ ways to assign timeslots to 6 people.

7. Find the number of ordered triplets of integers (x, y, z) such that

$$(x+y)^2 + (y+z)^2 + (z+x)^2 + (x+y)(x+z) + (y+z)(y+x) + (z+x)(z+y) = 328$$

Solution. The answer is $\boxed{96}$. Let $a = x+y, b = y+z, c = z+x$, and let $p = a+b, q = b+c, r = c+a$. Then, we obtain the equation $p^2 + q^2 + r^2 = 656$. Taking this equation modulo 4, we have that $p^2 + q^2 + r^2 \equiv 0 \pmod 4$. Since the square of an integer is congruent to either 0 or 1 modulo 4, we know that $p^2 \equiv q^2 \equiv r^2 \equiv 0 \pmod 4$, so p, q, r are divisible by 2. We can divide the equation by 4 and repeat this logic to find that p, q, r are divisible by 4. Next, note that we can write $a = \frac{r+p-q}{2}, b = \frac{p+q-r}{2}, c = \frac{q+r-p}{2}$, and we can write something similar for x, y, z. This means that for a given solution (p, q, r), there is a unique corresponding triple (x, y, z). Furthermore, since p, q, r are divisible by 4, we know that a, b, c are divisible by 2. This implies that x, y, z are integers, so (x, y, z) is a valid, unique solution. Thus, the number of solutions (x, y, z) is equal to the number of integer solutions (p, q, r) of $(\frac{p}{4})^2 + (\frac{q}{4})^2 + (\frac{r}{4})^2 = 41$. After some light computation, we see that there are 96 such solutions.

8. Find the greatest positive constant λ such that $x^5 + \frac{1}{x^5} - 2 \geq \lambda(x + \frac{1}{x} - 2)$ for all $x > 0$.

Solution. The answer is $\boxed{25}$. Let $y = x + \frac{1}{x} \geq 2$ for all $x > 0$. Then we can calculate

$$\begin{aligned}
x^2 + \frac{1}{x^2} &= y^2 - 2 \\
x^3 + \frac{1}{x^3} &= y^3 - 3x \\
x^4 + \frac{1}{x^4} &= y^4 - 4y^2 + 2 \\
x^5 + \frac{1}{x^5} &= y^5 - 5y^3 + 5y
\end{aligned}$$

The condition can be rewritten as

$$y^5 - 5y^3 + (5 - \lambda)y + 2(\lambda - 1) \geq 0 \qquad \forall y \geq 2$$

Since we know that equality occurs when $x = 1 \Rightarrow y = 2$, the derivative of the left hand side is equal to 0 when $y = 2$. This gives us $\lambda = 25$. We verify that this is valid:

$$y^5 - 5y^3 - 20y + 48 = (y - 2)^2(y + 3)(y^2 + y + 4) \geq 0 \qquad \forall y \geq 2$$

9. Eric and Julia like to spend their nights factoring. Last night they factored 2023. It's just $7 \cdot 17^2$, but everyone knows that! Tonight, they have a trickier problem in store. What's the smallest divisor of $2023^4 + 9^4$ that is greater than 2?

Solution. Observe that

$$2023^4 + 9^4 \equiv 3^4 + 1^4 \equiv 2 \pmod{4},$$

so the largest power of 2 that divides $2023^4 + 9^4$ is 2. With this, we know that our answer is the smallest odd prime factor of $2023^4 + 9^4$. Let p be an odd prime that divides $2023^4 + 9^4$. Then

$$2023^4 \equiv -9^4 \pmod{p}.$$

Let n be the smallest positive integer such that $2023^n \equiv -9^n \pmod{p}$. Squaring both sides of $2023^4 \equiv -9^4 \pmod{p}$ gives us

$$2023^8 \equiv 9^8 \pmod{p}.$$

Then by the minimality of n, n must divide 8. But since $2023^4 \not\equiv 9^4 \pmod{p}$, n cannot divide 4. This forces $n = 8$. But we also know that $\varphi(p) = p - 1$, which gives us

$$2023^{p-1} \equiv 1 \equiv 9^{p-1}.$$

Then using the minimality of n again, we see that $n \Big| p - 1$. Substituting, we get

$$p \equiv 1 \pmod{8}.$$

Thus, we only have to check primes that are $1 \pmod 8$. The first couple are

$$17, 41, 73, 89, 97, \ldots$$

We know 17 doesn't work, since 2023 is a multiple of 17 but 9 is not. Now we check 41. We can compute

$$2023^4 + 9^4 \equiv 14^4 + 9^4 \equiv 196^2 + 81^2 \equiv (-9)^2 + (-1)^2 \equiv 81 + 1 \equiv 0 \pmod{41}.$$

Hence, our desired prime is $\boxed{41}$.

10. In the variant *duck chess*, a rubber duck is a piece that can block a pair of rooks from attacking each other by acting as a barrier. On a 5×5 chessboard, or grid of squares, we can place exactly one piece on each square. Let a configuration of six rooks and two rubber ducks on the chessboard be *valid* if it satisfies the following:

    (a) With the six rooks and two rubber ducks placed, no pair of rooks attack each other

    (b) If the rubber ducks are removed, then there exists a rook such that if we remove it, there exists no pair of rooks that attack each other amongst the remaining five

    How many *valid* configurations are there? An example of one is shown below. (Note: A *rook* is a piece that is said to attack a square if it is on the same row or column and there exist no pieces between the rook and the tile)

    Solution. We solve this problem for general a general $n \times n$ board. We first consider the placements of our rooks, then our rubber ducks. To meet the second condition, note that five rooks do not attack each other if they are on distinct rows and columns. Any sixth rook we place will be attacked by a rook on the row and a rook on the column, forming a rectangle with one empty vertex. We thus must place our two rubber ducks such that they block the attacks on this rook from the row and the column. We attempt to count all possible placements of these rooks and ducks.

    Pick three distinct rows and three distinct columns on the grid (which can be done in $\binom{n}{3}^2$ ways). The outermost rows and columns determine the vertices of a rectangle. We can place three rooks on the vertices of this rectangle in 4 ways (choose which corner to leave empty). Then, the middle row and column uniquely determine the positions of where we place our rubber duck to block the attacks on one rook from the other two.

    Note that there are $n-2$ rooks remaining to place on the rest of the board. This is equivalent to placing $n-2$ rooks such that they don't attack each other on an $n-2 \times n-2$ grid, which can be done in $(n-2)!$ ways, as they must be on distinct rows and columns.

    Thus, for general n, our answer is

$$4 \cdot \binom{n}{3}^2 \cdot (n-2)!$$

For $n = 5$, this yields

$$4 \cdot \binom{5}{3}^2 \cdot 3! = \boxed{2400} \text{ configurations}$$