Tiebreaker Round Solutions

DMM 2023

1 Tiebreaker

1. Consider the polynomial
$$p(x) = x^3 + x^2 - 1.$$
Let r_1, r_2, and r_3 be the roots of of $p(x)$. Find the value of
$$r_1^4 + r_2^4 + r_3^4.$$

Solution. We rearrange the expression into
$$(r_1^4 - 1) + (r_2^4 - 1) + (r_3^4 - 1) + 3.$$
This allows us to use the fact
$$x^4 - 1 = (x-1)(x^3 + x^2 + x + 1).$$
Note that the roots r_i satisfy
$$r_i + 2 = r_i^3 + r_i^2 - 1 + (r_i + 2) = r_i^3 + r_i^2 + r_i + 1.$$
Then our desired quantity is
$$\begin{aligned}(r_1^4 - 1) + (r_2^4 - 1) + (r_3^4 - 1) + 3 &= (r_1 - 1)(r_1 + 2) + (r_2 - 1)(r_2 + 2) + (r_3 - 1)(r_3 + 2) + 3 \\ &= r_1^2 + r_1 - 2 + r_2^2 + r_2 - 2 + r_3^2 + r_3 - 2 + 3 \\ &= (r_1^2 + r_2^2 + r_3^2) + (r_1 + r_2 + r_3) - 3.\end{aligned}$$
Using Vieta's formulas, we know
$$\begin{aligned} r_1 + r_2 + r_3 &= -\frac{1}{1} = -1 \\ r_1r_2 + r_1r_3 + r_2r_3 &= \frac{0}{1} = 0.\end{aligned}$$
This also lets us determine
$$r_1^2 + r_2^2 + r_3^2 = (-1)^2 - 2(0) = 1.$$
Thus, our desired answer is
$$(1) + (-1) - 3 = \boxed{-3}$$

2. Calvin has 10 courses he wishes to take, each numbered 1,2,. . . ,10. However, course 1 serves as a prerequisite to course 2, course 3 is a prerequisite for course 4, etc.. In how many different orders can Calvin take the courses given that he can only take one course at a time and that the prerequisite conditions must be satisfied for all courses (i.e. course 1 must be taken before course 2)?

   Please answer with a specific number, not a formula.

Solution. First, we think about ordering the courses with no restrictions, for which there are 10!. To account for our ordering restrictions, note that for the restriction of Course 1 before Course 2, exactly half of the permutations have this ordering. The five restrictions are independent of each other, thus our answer is $10!/2^5 = \boxed{113400}$

3. You have four 6-sided dice. One die has $\sin^2(e)$ on half of the faces and 0 on the other half. The other 3 dice are identical, each has $\cos^2(e)$ on one face and 0 on every other face. You roll each die once, what is the expected value of the sum of all the dice?

The expected value of the first die is $\frac{1}{2}\sin^2(e)$. The expected value of one of the other three dice is $\frac{1}{6}\cos^2(e)$. Thus, our answer is

$$\frac{1}{2}\sin^2(e) + 3 \cdot \frac{1}{6}\cos^2(e) = \frac{1}{2}\left(\cos^2(e) + \sin^2(e)\right) = \boxed{\frac{1}{2}}$$