2024 DUKE MATH MEET GUTS ROUND SOLUTIONS

Set 1.

1. The set of letters has

$$\mathrm{U}^2,\ \mathrm{E}^2,\ \mathrm{I}^2,\ \text{and 8 distinct singles} \quad \text{(total 14 letters)}.$$

All arrangements: $\frac{14!}{2!\,2!\,2!} = \frac{14!}{8}$. If the two U's are adjacent (treat UU as one block), we have 13 items with multiplicities E^2, I^2, so $\frac{13!}{2!\,2!} = \frac{13!}{4}$. Thus the desired count is

$$\frac{14!}{8} - \frac{13!}{4} = \frac{a!}{8} - \frac{b!}{4} \quad \Rightarrow \quad a = 14,\ b = 13,\ a + b = \boxed{27}.$$

2. Compute $\left(\log 2000\right)^{\log\log 200} - \left(\log 200\right)^{\log\log 2000}$ (base 10). Using $a^{\log b} = b^{\log a}$ for logs with the same base,

$$\left(\log 2000\right)^{\log\log 200} = (\log 200)^{\log\log 2000},$$

so the difference is $\boxed{0}$.

3. Largest power of 3 dividing a_{2024} for $a_n = na_{n-1},\ a_1 = 1$. We have $a_n = n!$, so the exponent of 3 in 2024! is

$$\begin{aligned}\nu_3(2024!) &= \left\lfloor \frac{2024}{3} \right\rfloor + \left\lfloor \frac{2024}{9} \right\rfloor + \left\lfloor \frac{2024}{27} \right\rfloor + \left\lfloor \frac{2024}{81} \right\rfloor + \left\lfloor \frac{2024}{243} \right\rfloor + \left\lfloor \frac{2024}{729} \right\rfloor \\ &= 674 + 224 + 74 + 24 + 8 + 2 = \boxed{1006}.\end{aligned}$$

Hence the largest power of 3 dividing a_{2024} is $\boxed{3^{1006}}$.

Set 2.

1. We want to prove that

$$f(n) = 2^{n-1} \quad \text{for all } n \geq 1.$$

Base Case: For $n = 1$ and $n = 2$:

$$f(1) = 1 = 2^{1-1} = 2^0, \qquad f(2) = 2 = 2^{2-1} = 2^1.$$

Inductive Step: Assume the induction hypothesis

$$f(n) = 2^{n-1} \quad \text{and} \quad f(n-1) = 2^{(n-1)-1} = 2^{n-2}.$$

Substituting these into the recurrence yields

$$f(n+1) = 2^{n-1} + 2 \cdot 2^{n-2}.$$

Simplifying,

$$f(n+1) = 2^{n-1} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^n = 2^{(n+1)-1}.$$

Thus, the statement holds for $n + 1$. It follows that our answer for this question is

$$\boxed{2023}.$$

2. Suppose the two numbers rolled are a and b. Andrew's product then is ab, while Brandon's sum is $a+b$. We want to compute the probability that $ab > a+b$.

   To solve this inequality, we move the sum $a+b$ to the left hand side, where we use Simon's Favorite Factoring Trick to factorize and get:

$$(a-1)(b-1) > 1$$

   We know that the inequality $(a-1)(b-1) \geq 1$ holds for all $a, b \geq 2$ as a, b are integer values from 1 to 6 and $(a-1) \geq 1$ for $a \geq 2$. Thus we have $5 \cdot 5 = 25$ pairs of (a, b) total. But from here we have to subtract the case where $(a-1)(b-1) = 1$, or namely, $(a, b) = (2, 2)$, through which we get $25 - 1 = \boxed{24}$

3. Let the side length of S_1 be s. Then the diameter of C_1 is s. The area of the shaded region inside S_1 and outside C_1 is

$$s^2 - \pi\left(\frac{s}{2}\right)^2 = s^2 - \frac{\pi s^2}{4} = s^2\left(1 - \frac{\pi}{4}\right).$$

   If the diameter of C_1 is s, then the diagonal of S_2 is s, so the side length of S_2 (and hence the diameter of C_2) is $\frac{s}{\sqrt{2}}$. Therefore, the shaded area inside S_2 and outside C_2 is

$$\left(\frac{s}{\sqrt{2}}\right)^2 - \pi\left(\frac{s}{2\sqrt{2}}\right)^2 = \frac{s^2}{2} - \frac{\pi s^2}{8} = s^2\left(\frac{1}{2} - \frac{\pi}{8}\right).$$

   Notice this is exactly $\frac{1}{2}$ of the previous shaded area, and this ratio holds for any two consecutive shaded regions $S_k \setminus C_k$ and $S_{k+1} \setminus C_{k+1}$. Normalize by taking $s = 1$ (so the probability of hitting S_1 is 1). The desired probability p is the sum of the shaded areas:

$$p = \left(1 - \frac{\pi}{4}\right) + \frac{1}{2}\left(1 - \frac{\pi}{4}\right) + \left(\frac{1}{2}\right)^2\left(1 - \frac{\pi}{4}\right) + \cdots = \frac{1 - \frac{\pi}{4}}{1 - \frac{1}{2}} = 2 - \frac{\pi}{2}.$$

   Thus $p = \boxed{2 - \frac{\pi}{2}}$. In the form $a - \frac{b\pi}{c}$ we have $a = 2,\ b = 1,\ c = 2$, so $a + b + c = \boxed{5}$.

Set 3.

1. We will show that this equation has only one such solution $(k, n) = (0, 0)$.

   For $k \geq 1$: $2024 \equiv 2 \pmod 3$. Thus

$$2024^k - 1 \equiv 2^k - 1 \pmod 3.$$

   It follows that $2024^k - 1$ is divisible by 3 if k is even and not divisible by 3 if k is odd.

   However, since $2024^k - 1$ is odd for any k, it follows that the corresponding n that gives a solution must also be odd. Thus $\frac{n+1}{k+1}$ cannot be an integer, as the denominator will be odd and the numerator will be even. So, the answer is $\boxed{0}$.

2. Let $f(x) = x^3 - 11x^2 + 56x - 6$ and set P such that

$$P = \left(1 + \frac{a^2}{b^2}\right)\left(1 + \frac{b^2}{c^2}\right)\left(1 + \frac{c^2}{a^2}\right)$$

We can multiply both sides by $(a^2b^2c^2)$ to get:

$$36P = (a^2b^2c^2)P = (a^2b^2c^2) \cdot \left(1 + \frac{a^2}{b^2}\right)\left(1 + \frac{b^2}{c^2}\right)\left(1 + \frac{c^2}{a^2}\right) = (b^2 + a^2)(c^2 + b^2)(a^2 + c^2)$$

with the first equality following from Vieta's and the last from distributing b^2, c^2, and a^2 to each parenthesis term, respectively.

Now let's consider the value $a^2+b^2+c^2$, which we can compute as $(a+b+c)^2-2(ab+bc+ca) = 121 - 2 \cdot 56 = 9$ from Vieta's. Thus, we can rewrite the above expression as:

$$36P = (9 - c^2)(9 - a^2)(9 - b^2) = (3 - a)(3 - b)(3 - c)(3 + a)(3 + b)(3 + c)$$

which follows from difference of squares. We also find that since $f(x)$ is a monic polynomial, $f(x) = (x - a)(x - b)(x - c)$, so we can rewrite:

$$f(3) = (3 - a)(3 - b)(3 - c)$$

$$f(-3) = (-3 - a)(-3 - b)(-3 - c) = (-1)^3(3 + a)(3 + b)(3 + c)$$

Thus, $36P = -f(3)f(-3)$

$$\longrightarrow 36P = -(3^3 - 11 \cdot 3^2 + 56 \cdot 3 - 6)((-3)^3 - 11 \cdot (-3)^2 + 56 \cdot (-3) - 6$$

$$\longrightarrow 36P = 27000$$

$$P = 750$$

There is only one value of the product and it is $\boxed{750}$

3. First, note that the x are all third roots of unity rotated counterclockwise $30°$, and the y are all fourth roots of unity. We can draw the following diagram, where

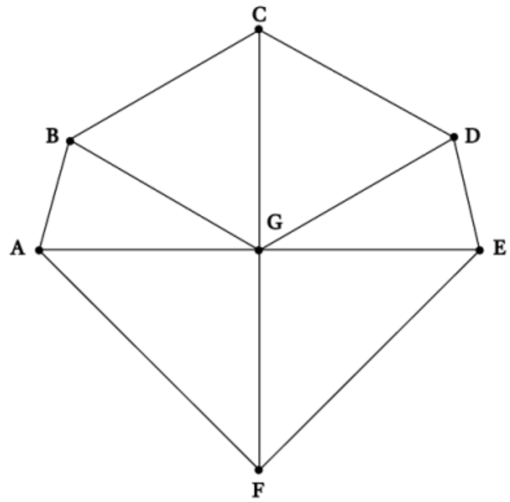
$$AG = BG = CG = DG = EG = FG = 1.$$

$[AEF]$ is clearly 1, since $AE = 2$, $GF = 1$, and $\angle AGF$ is right.

Next, $\triangle BGC$ and $\triangle CGD$ are equilateral with side length 1, so each has area $\frac{\sqrt{3}}{4}$, for a total of $\frac{\sqrt{3}}{2}$.

Finally,

$$[ABG] = [EDG] = \tfrac{1}{2} \cdot AG \cdot BG \cdot \sin(\angle BGA).$$



Since $\angle BGA = 30°$, we get $[ABG] = [EDG] = \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin 30° = \frac{1}{4}$, for a total of $\frac{1}{2}$.

Adding the pieces,

$$1 + \frac{1}{2} + \frac{\sqrt{3}}{2} = \boxed{\frac{3}{2} + \frac{\sqrt{3}}{2}}.$$

Set 4.

1. Because $ABCD$ is inscribed in a circle and $\angle CDA = 120°$, we have $\angle ABC = 60°$. Because of this and the fact that $\triangle ABC$ is isosceles, $\triangle ABC$ is in fact equilateral, so $AC = 6$. Using the law of cosines on $\triangle ACD$,

$$6^2 = 16 + CD^2 + 4CD \Rightarrow CD^2 + 4CD - 20 = 0.$$

Solving for CD, we get $CD = 2\sqrt{6} - 2$. From Ptolemy's Theorem, $6BD = 6(2\sqrt{6} - 2) + 6(4)$, so

$$BD = 2\sqrt{6} + 2 \to \boxed{10}.$$

2. We notice that the abc term in the denominator is not symmetric with respect to all a, b, c, d. Thus, we multiply the fraction by $\frac{d}{d}$ to introduce symmetry[1] and get

$$\frac{d}{-36 - d(d-1)^2 + 36d} = \frac{-d}{d^3 - 2d^2 - 35d + 36}$$

From here, we notice that the constant term in the denominator is the same as the constant term in the initial polynomial (except for the sign). This motivates us to multiply the fraction again by $\frac{(d-1)}{(d-1)}$

$$\frac{-d(d-1)}{d^4 - 3d^3 - 33d^2 + 71d - 36} = \frac{-d(d-1)}{-8d} = \frac{d-1}{8}$$

[1]Introducing the d term, and thus asymmetry, to the numerator is fine as symmetry in the demominator takes precedence over symmetry in the numerator. As long as the denominator stays constant for our cyclic sum, the numerator is cleared up by the cyclic sum: e.g. $\sum_{\text{cyc}} \frac{a}{x} = \frac{a}{X} + \frac{b}{X} + \frac{c}{X} + \frac{d}{X} = (a + b + c + d)\frac{1}{X}$

The first equality follows from the fact that since a, b, c, d are roots of the polynomial, $d^4 - 3d^3 - 33d^2 + 79d - 36 = 0$, or $d^4 - 3d^3 - 33d^2 + (71+8)d - 36 = 0$, leading to the above equality after subtracting $8d$ from both sides.

$$\sum_{\text{cyc}} \frac{d-1}{8} = \frac{a+b+c+d-4}{8} = \frac{3-4}{8} = -\frac{1}{8}$$

Thus the answer is $1 + 8 = \boxed{9}$

3. Let V_B be the volume of the box, V_T the volume of each tennis ball, and N the initial number of balls. The total volume of the balls is NV_T, and the volume of the empty space is $V_B - NV_T$. Given the ratio $1 : k$, we have

$$\frac{NV_T}{V_B - NV_T} = \frac{1}{k} \quad \Rightarrow \quad V_B = (k+1)NV_T.$$

Now let P be the number of balls removed, where P is prime. The remaining balls have volume $(N-P)V_T$, and the empty space becomes $V_B - (N-P)V_T$. With the new ratio $1 : k^2$, we get

$$\frac{(N-P)V_T}{V_B - (N-P)V_T} = \frac{1}{k^2}.$$

Substituting $V_B = (k+1)NV_T$ gives

$$\frac{N-P}{(k+1)N - (N-P)} = \frac{1}{k^2} \quad \Longrightarrow \quad k^2(N-P) = kN + P.$$

Rearranging,

$$N = \frac{P(k^2+1)}{k^2-k}.$$

Since N is an integer, $P \mid (k^2+1)$. Because P is prime and $\gcd(k, k^2+1) = 1$, we conclude $k = P$. Thus

$$N = \frac{P^2+1}{P-1} = P + 1 + \frac{2}{P-1}.$$

For N to be an integer, $P - 1 \mid 2$, so $P = 2$ or $P = 3$, and in either case $N = 5$.

Therefore, the initial number of balls in the container is

$$\boxed{5}.$$

Set 5.

1. Let $\frac{9n+10}{4n+17} = \frac{a^2}{b^2}$ for some relatively prime positive integers a and b. Then $9n + 10 = ka^2$ and $4n + 17 = kb^2$ for some integer k. Then, multiplying each equation by 4 and 9 respectively, gives $36n + 40 = 4ka^2$ and $36n + 153 = 9kb^2$. Subtracting the two equations gives, $113 = 9kb^2 - 4ka^2 = k(3b + 2a)(3b - 2a)$. Since 113 is prime, two of these factors must be 1 or -1,

and the other factor must be 113 or -113. But since a and b are positive integers, $3b+2a$ cannot be 1 and it cannot be negative. Thus, $3b+2a=113$. Then $3b-2a$ is 1 or -1. If $3b-2a=-1$, then adding the two equations gives $6b=112$, but since 112 is not a multiple of 6, this is impossible. Thus, $3b-2a=1$, so $6b=114$ and $b=19$. We also see that $a=28$ and $k=1$. Plugging in these values gives $n=\boxed{86}$ as our only solution.

2. We seek positive integers (a,b,c) with

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=2.$$

Note first that none of a,b,c can be 1 (otherwise the product of the other two factors would be 1, impossible for positive integers). If $a,b,c\geq 4$, then

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)\leq\left(1+\frac{1}{4}\right)^3=\left(\frac{5}{4}\right)^3<2,$$

so at least one of a,b,c is ≤ 3. Thus some variable is 2 or 3.

Case 1: One variable equals 2. WLOG set $b=2$. Then

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{c}\right)=\frac{4}{3}\iff 3(ac+a+c+1)=4ac\iff ac-3a-3c=3\iff(a-3)(c-3)=12.$$

Positive factor pairs of 12 yield

$$(a,c)=(4,15),(5,9),(6,7)\text{ and swaps,}$$

so with $b=2$ there are 6 ordered triples. Placing the 2 in any of the three positions contributes $3\cdot 6=18$ solutions.

Case 2: No variable equals 2*, so some variable equals* 3. WLOG set $b=3$. Then

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{c}\right)=\frac{3}{2}\iff 2(ac+a+c+1)=3ac\iff ac-2a-2c=2\iff(a-2)(c-2)=6.$$

Positive factor pairs of 6 yield

$$(a,c)=(3,8),(4,5)\text{ and swaps.}$$

Thus the unordered multisets are $\{3,3,8\}$ and $\{3,4,5\}$, contributing $\frac{3!}{2!}=3$ and $3!=6$ ordered triples, respectively; total 9.

Adding both cases, the number of ordered triples is

$$18+9=\boxed{27}.$$

3. Define

$$f(n) = \begin{cases} n/2, & n \text{ even}, \\ n+1, & n \text{ odd}, \end{cases} \qquad g(x) = \min\{k : f^k(x) = 1\}, \quad g(1) = 0.$$

From the definition we get the recurrences

$$g(2n) = 1 + g(n), \qquad g(2n+1) = 1 + g(2n+2) = 2 + g(n+1).$$

Let $S_m = \sum_{x=1}^{2^m} g(x)$. Split into evens and odds (and note $g(1) = 0$):

$$\begin{aligned} S_m &= \sum_{n=1}^{2^{m-1}} g(2n) \ + \ \sum_{n=1}^{2^{m-1}} g(2n-1) \\ &= \sum_{n=1}^{2^{m-1}} \big(1 + g(n)\big) + \Big[g(1) + \sum_{n=2}^{2^{m-1}} \big(2 + g(n)\big)\Big] \\ &= 2S_{m-1} + 3 \cdot 2^{m-1} - 2, \end{aligned}$$

with $S_0 = g(1) = 0$.

Solve the linear recurrence: try a particular solution of the form $S_m^{(p)} = a\,m\,2^m + b$. Then

$$S_m^{(p)} - 2S_{m-1}^{(p)} = a2^m - b \stackrel{!}{=} \tfrac{3}{2}2^m - 2,$$

so $a = \frac{3}{2}$, $b = 2$. Hence the general solution is

$$S_m = \left(\tfrac{3}{2}m + c\right) 2^m + 2,$$

and $S_0 = 0$ gives $c = -2$. Thus

$$S_m = \Big(\tfrac{3}{2}m - 2\Big)2^m + 2.$$

Finally, with $2^6 = 64$,

$$\sum_{x=1}^{64} g(x) = S_6 = \Big(\tfrac{3}{2} \cdot 6 - 2\Big)2^6 + 2 = (9-2) \cdot 64 + 2 = \boxed{450}.$$

Set 6.

1. $\boxed{40,000}$
2. $\boxed{17,000,000}$
3. $\boxed{47,600}$