2024 DUKE MATH MEET INDIVIDUAL ROUND

Problems 1-2 **Name** ________________

Time Limit: 10 minutes **Team** ________________

Problem 1 5 distinct positive integers are written on the board. Every second a computer selects two of the integers and replaces them with their average. Eventually, only one number is left: 11. Given this, what is the largest possible value of the minimum of the five integers at the beginning, among all such possibilities?

Problem 2 Consider the eight points $(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3)$. These points define $\binom{8}{2}$ line segments. Suppose we choose two of these segments uniformly at random. Let the x-coordinate of the intersection of the lines (if an intersection exists) be x'. The probability that x' exists and that $0 < x' < 1$ can expressed as $\frac{a}{b}$, where $\gcd(a,b) = 1$. Find $a + b$.

ANSWER TO PROBLEM 1

ANSWER TO PROBLEM 2

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Problems 3-4 **Name** ____________________

Time Limit: 10 minutes **Team** ____________________

Problem 3 Let $\triangle ABC$ be a triangle with angle bisector AD such that the area of $\triangle ABC$ is 12, $AC = 8$, $AD = 5$, and $AB = 6$. What is the length of BD^2?

NOTICE: Problem 3 was voided in-contest. All contestants were awarded 1 point for this problem.

Problem 4 Harry and Chris are playing a game on a 100 by 100 square grid of tiles. Chris will choose 210 of the tiles to paint red, and then Harry will pick a path from the bottom left tile of the grid to the top right tile of the grid based on Chris's selection. For the sake of this problem, a "path" is a sequence of tiles such that every tile is adjacent (horizontally or vertically, but not diagonally) to the tile preceding it. Chris's goal is to maximize the number of red tiles in Harry's path, while Harry's goal is to minimize the same quantity. Given that Chris picks the red tiles optimally, what is the minimum number of red tiles Harry's path can contain?

ANSWER TO PROBLEM 3

ANSWER TO PROBLEM 4

2024 DUKE MATH MEET INDIVIDUAL ROUND

Problems 5-6 **Name** ________________

Time Limit: 10 minutes **Team** ________________

Problem 5 How positive integers exist with three not necessarily distinct digits abc satisfying $a \neq 0$ and $c \neq 0$ such that abc and cba are divisible by 4?

Problem 6 Given that there exist real numbers a, b, c, d such that $4b^2c^2 + 16b^2d^2 + 16a^2c^2 + 64a^2d^2 = 31$ and $ac + bd = \frac{\sqrt{31}}{4}$, compute $\frac{bc}{ad}$.

ANSWER TO PROBLEM 5

ANSWER TO PROBLEM 6

2024 DUKE MATH MEET INDIVIDUAL ROUND

Problems 7-8 **Name** ____________________

Time Limit: 10 minutes **Team** ____________________

Problem 7 Consider the following partially filled-in grid. A *galactic* grid is a grid such that, for any pair of rows and pair of columns, the four squares determined by their intersections have an even sum. How many ways are there to fill in the blank entries in the following grid with either a 0 or 1 such that the grid is *galactic*?

			0	1
	1		0	
			1	0
	0			

Problem 8 Find the value of

$$\sum_{m=0}^{6}\sum_{n=0}^{6}\binom{12-m-n}{6-m}\binom{m+n}{m}$$

ANSWER TO PROBLEM 7

ANSWER TO PROBLEM 8

2024 DUKE MATH MEET INDIVIDUAL ROUND

Problems 9-10 **Name** ________________

Time Limit: 10 minutes **Team** ________________

Problem 9 Let $f(x) = \frac{x}{\sqrt{x^2-1}}$. Define $f^n(x) = f(f(f \cdots (f(x)) \cdots))$ with f applied recursively n times. The value $f^{2024}(2024)$ can be written as $\frac{a\sqrt{b}}{c}$. Find $a + b + c$.

Problem 10 Triangle ABC has side lengths $AB = 17$, $BC = 25$, and $CA = 28$. Point D lies on $\overline{BC}$ such that $AD \perp BC$. Point E lies on $\overline{AC}$ such that $BE \perp AC$. Let P be the point on $\overline{AC}$ such that $\angle BPC = 180^\circ - \angle BAC$. Let Q be the intersection of lines DE and BP. The circumradius of triangle PQE can be expressed as $\frac{a}{b}$, where $\gcd(a, b) = 1$. Find $a + b$.

ANSWER TO PROBLEM 9 **ANSWER TO PROBLEM 10**