Individual Solutions

1. Notice that the sum of the five numbers at any second remains the same. Therefore, the sum of the original five values is $5 \cdot 11 = 55$. Let a be the smallest of the five original values. Then their sum is at least

$$a + (a+1) + \cdots + (a+4) = 5a + 10.$$

To maximize a, we want $55 = 5a + 10$, which gives $a = \boxed{9}$.

2. There are $\binom{8}{2} = 28$ total line segments, and there are $\binom{28}{2}$ total ways to pick two of these segments. These two segments share a point in common with an x-coordinate on the interval $(0, 1)$ if and only if one point from each line lies on the lines $x = 0$ and $x = 1$, and the two lines cross each other. This occurs exactly once for each set of 4 points such that 2 points lie on the lines $x = 0$ and 2 points lie on the line $x = 1$. Thus there are $\binom{4}{2}^2$ total intersections. Therefore, the desired probability is $\frac{\binom{4}{2}^2}{\binom{28}{2}} = \boxed{\frac{2}{21}}$

3. VOIDED

4. To ease discussion, give each tile in the grid a pair of (x, y) coordinates. The bottom left tile is $(1, 1)$ and the top right tile is $(100, 100)$. Now consider the top-left to bottom-right diagonals, which are defined by

$$x + y = n$$

for some integer n. Note that a path from the bottom left tile to the top right tile will transition from diagonals, starting with the $n = 2$ diagonal, moving to the $n = 3$ diagonal, and continuing until the $n = 200$ diagonal. So, Chris's optimal strategy is to color entire diagonals. Coloring the smallest ones first, Chris has just enough to color the diagonals for $n = 1, 2, \ldots, 14$ and $187, 188, \ldots, 200$. Hence, Harry is forced to use at least 28 red tiles.

5. A three-digit number is divisible by 4 iff its last two digits form a multiple of 4. Thus we need

$$10b + c \equiv 0 \pmod{4} \quad \text{and} \quad 10b + a \equiv 0 \pmod{4}.$$

Since $10 \equiv 2 \pmod{4}$, these are equivalent to

$$c \equiv -2b \pmod{4}, \qquad a \equiv -2b \pmod{4}.$$

Hence $a \equiv c \equiv -2b \pmod{4}$.

Case 1: b even. Then $2b \equiv 0 \pmod{4}$, so $a, c \equiv 0 \pmod{4}$. Because $a \neq 0$ and $c \neq 0$, we must have $a, c \in \{4, 8\}$: 2 choices for a and 2 for $c \Rightarrow 4$ pairs for each of the 5 even b's $(0, 2, 4, 6, 8)$.

Case 2: b odd. Then $2b \equiv 2 \pmod{4}$, so $a, c \equiv 2 \pmod{4}$. Thus $a, c \in \{2, 6\}$: again 4 pairs for each of the 5 odd b's $(1, 3, 5, 7, 9)$.

Therefore the total number of such integers is

$$5 \cdot 4 + 5 \cdot 4 = \boxed{40}.$$

6. Note that

$$4b^2c^2 + 16b^2d^2 + 16a^2c^2 + 64a^2d^2 = (16a^2 + 4b^2)(c^2 + 4d^2) = 31,$$

and dividing both sides by 4 gives

$$(4a^2 + b^2)(c^2 + 4d^2) = \frac{31}{4}.$$

By the Cauchy–Schwarz inequality,

$$(4a^2 + b^2)(c^2 + 4d^2) \geq (2ac + 2bd)^2.$$

However, we are given $ac + bd = \frac{\sqrt{31}}{4}$, so $2(ac + bd) = \frac{\sqrt{31}}{2}$ and hence $(2ac + 2bd)^2 = \frac{31}{4}$. Thus we have equality, so the vectors $(2a, b)$ and $(c, 2d)$ are proportional; in particular,

$$\frac{2d}{b} = \frac{c}{2a}.$$

Cross-multiplying and rearranging yields $4ad = bc$, i.e.

$$\frac{bc}{ad} = \boxed{4}.$$

7. We first fill in the tiles on the grid that we can uniquely determine, which are the numbers i n **bold** below. Then, note that placing one tile uniquely determines a set of other tiles. In particular, we can test these dependencies by putting in a test number and seeing which tiles get determined. We call such a set of tiles a component, which is defined as one tile determining the rest of the tiles in the component uniquely. These components are labeled X, Y, Z in the grid. Since these are the three variable elements in our grid, there are $2^3 = \boxed{8}$ such grids.

X	**1**	Y	0	1
X	1	Y	0	**1**
Z	Z	Z	Z	Z
X	**0**	Y	1	0
X	0	Y	**1**	**0**

8. The crux of the problem is a counting argument. Assume we are on a square grid, with bottom left corner $(0, 0)$ and top right corner (6,6). Then

$$\binom{12 - m - n}{6 - m}\binom{m + n}{m}$$

is the number of paths from $(0, 0)$ to $(6, 6)$ that pass through (m, n). Then, we are summing this quantity over all points (m, n) in the grid.

We can interpret this summation differently. Note that every path from $(0,0)$ to $(6,6)$ contributes a value of 13 to the summation, once for each point on the path. So, the summation is 13 times the total number of paths from $(0,0)$ to $(6,6)$ which is

$$13\binom{12}{6} = 13 \cdot 924 = \boxed{12012}.$$

9. Writing out $f(f(x))$:

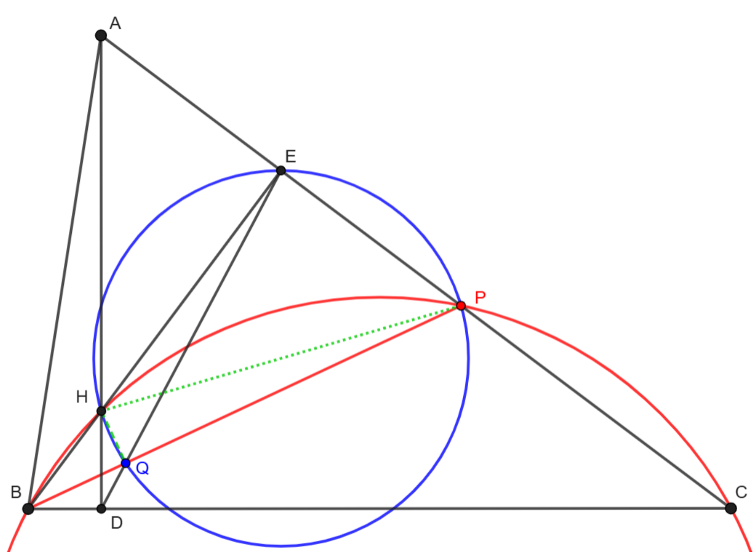
$$f(f(x)) = \frac{\frac{x}{\sqrt{x^2-1}}}{\sqrt{\left(\frac{x}{\sqrt{x^2-1}}\right)^2 - 1}} = \frac{x}{\sqrt{x^2-1}} \cdot \frac{1}{\sqrt{\frac{x^2}{x^2-1} - 1}}$$

$$= \frac{x}{\sqrt{x^2-1}} \cdot \frac{1}{\sqrt{\frac{x^2}{x^2-1} - \frac{x^2-1}{x^2-1}}} = \frac{x}{\sqrt{x^2-1}} \cdot \frac{1}{\sqrt{\frac{1}{x^2-1}}} = x.$$

Therefore, applying this function an even number n of times yields $f^n(x) = x$. So the answer in the required form is $\frac{2024\sqrt{1}}{1}$, and hence the value is

$$\boxed{2026}.$$

10. In the diagram below, we've added point H, the orthocenter of ABC, to aid in the solution.



It is well-known that

$$\angle BHC = 180^\circ - \angle BAC = \angle BPC,$$

so $BHPC$ is cyclic. Then focusing on triangle BPC, we see that DE is the Simson line of H with respect to BPC. This means $Q = BP \cap DE$ is the foot of H onto BP, so $\angle HQP = 90^\circ$.

Then since $\angle HEP = 90^\circ$ as well, we see $HEPQ$ is cyclic with diameter HP. So, our desired circumradius is half the length of HP.

To compute HP, we first observe that

$$\angle BPA = 180 - \angle BPC = \angle BAC,$$

so triangle BPA is isosceles with $BA = BP$. This means BE is the perpendicular bisector of AP, and hence by symmetry we have $HP = HA$. It can be shown that $AH = 2R\cos A$, so our desired quantity is $R\cos A$. It remains to compute it.

For $\cos A$, we use law of cosines to find

$$\cos A = \frac{17^2 + 28^2 - 25^2}{2(17)(28)} = \frac{8}{17}.$$

We then have $\sin A = \frac{15}{17}$. Hence, by law of sines, we know

$$R = \frac{25}{2 \cdot \frac{15}{17}} = \frac{85}{6}.$$

Thus, our final answer is

$$\frac{85}{6} \cdot \frac{8}{17} = \boxed{\frac{20}{3}}.$$