Permutations!

DMM Power Round 2024

For questions asking you to **find**, **evaluate**, **give**, or **compute**, you do not need to give any additional justification, and there are no partial credits available for wrong solutions. For questions asking you to **show** or to **prove**, in order to receive full credits you should show a concrete, precise proof, but partial credits are available for these questions.

There are **50 points** in total, and the point value of each question is written beside the problem number.

1 The Symmetric Group (16 points)

In what follows, let n be a positive integer, and $[n] := \{1, \cdots, n\}$

Definition 1. *We define a* ***permutation*** *of* $[n]$ *to be a bijection* $[n] \to [n]$*. That is, it takes in a number in* $[n]$ *and outputs a unique number in* $[n]$*. Let* S_n *denote the set of permutations of* $[n]$*. It is called* ***the symmetric group on*** n ***elements****.*

There are two ways to think about and write a permutation:

- We can focus on the values: for example $4213 \in S_4$ denotes the permutation $\pi\colon 1 \mapsto 4, 2 \mapsto 2, 3 \mapsto 1, 4 \mapsto 3$. We call this **value notation**.
- OR we can focus on the cycles: we can consider an arrow $1 \to 2, 2 \to 1$ and $3 \to 3$. We realize that if we start at 1, we return to 1 in two steps, while if we start at 3 we return to it in one step. Thus, we can view it as a swap of 1 and 2, so we write this as (12)(3) or as just (12) since we leave out cycles of length 1 by convention.

  Note that this **cycle notation** is not necessarily unique. For instance, the permutation $4213 \in S_4$ can be written as (143), (431), or (314).

We can also combine permutations by composing them. More specifically, let π, ρ be permutations of $[n]$. Then their composition $\pi \circ \rho$ is a permutation satisfying $\pi \circ \rho(j) = \pi(\rho(j))$ for all $1 \leq j \leq n$. For example, $(345) \in S_5$ is 12453, and $(123) \circ (345) = 23451$.

We use id to denote the special **identity** permutation, which is the permutation that moves $i \mapsto i$ for every i. This has the property that $id \circ \pi = \pi \circ id = \pi$ for any permutation π, which is to say that id preserves other permutations.

Problem 1. [2] Evaluate $(12) \circ (23)$ and $(23) \circ (12)$ in S_3.

You should get that $(12) \circ (23) \neq (23) \circ (12)$. This shows that the composition of permutations is not commutative, so we cannot simply swap the order of permutations.

Next, there is a notion of inverting a permutation, which reverses the effect of the permutation.

Definition 2. *For every permutation* $\pi \in S_n$*, there is a unique permutation* $\rho \in S_n$ *such that* $\rho \circ \pi = id$*. We call* ρ *the* ***inverse*** *of* π*, and we denote it as* $\rho = \pi^{-1}$*.*

Problem 2. [2] Compute the inverse of the permutation 35421.

Problem 3. **[2]** Suppose that we have a cycle permutation $\pi = (c_1 c_2 \cdots c_k)$ of $[n]$ where $c_i \in [n]$ for each $i = 1, \dots, k$. Compute π^{-1} in terms of c_1 through c_k.

In general, if π contains multiple cycles, then π^{-1} is the permutation consisting of the inverse of each cycle.

Problem 4. **[2]** Show that if $\rho, \pi \in S_n$ satisfies $\rho \circ \pi = id$, then $\pi \circ \rho = id$.

Definition 3. *We say a* ***transposition*** *is a permutation that swaps two elements and does not change the other elements. For example,* (12) *is a transposition since it swaps* 1 *and* 2 *while preserving every other number. Similarly,* (23) *and* (13) *are also transpositions.*

Problem 5. **[4]** Give the following permutations in “value" form: for example, instead of writing $(12)(3)$ or (12), write 213.

i) (24)

ii) $(14) \circ (24)$

iii) (14)

iv) The permutation attained by first swapping the number in **positions** $1, 4$, then swapping the numbers in **positions** $2, 4$.

Problem 6. **[4]** Fix $k, n \in \mathbb{N}$. Let $s_1, \cdots, s_k$ be **transpositions** of S_n. We can multiply them $s_1 \circ s_2 \circ \cdots \circ s_k$. Let $s_i = (a_i b_i)$. Show that $s_1 \circ s_2 \circ \cdots \circ s_k$ will be attained by the following procedure:

- Start with the identity permutation.
- Swap numbers in positions a_1, b_1
- Then, swap numbers in positions a_2, b_2
- $\cdots$
- Eventually, swap numbers in positions a_k, b_k

(Hint: Induct on k; start by understanding the cases $k = 1, 2$ well, then show how to go from k to $k + 1$. The previous problem should give you a solid understanding of what happens when $k = 2$)

2 Cycle Type (10 points)

Definition 4. *We define the* ***cycle type*** *of a permutation* $\pi \in S_n$ *to be a sequence* $(c_i)_{i \geq 1}$*, where* c_i *is the number of cycles of length* i*.*

Note that $(c_i)_{i \geq 1}$ is eventually zero. We usually omit the eventually zero terms for brevity.

For example, the cycle type of the permutation $23154 = (123)(45) \in S_5$ is $[0, 1, 1]$, since we have no cycles of length 1, one cycle of length 2, and one cycle of length 3.

Problem 7. **[2]** Compute the number of permutations of S_5 with cycle type $[0, 1, 1]$.

Sometimes, for $d_1, d_2, \cdots, d_m \in [n]$, we write $(d_1, \cdots, d_m)$ for the cycle that takes d_i to d_{i+1} (where $d_{m+1} = d_1$). For example, (123) may be written as $(1, 2, 3)$. The commas are there to enhance readability.

Definition 5. *We say two permutations $\sigma, \tau \in S_n$ are* **conjugate** *if there exists some $\rho \in S_n$ such that $\sigma = \rho\tau\rho^{-1}$.*

Problem 8. [3] Show that if τ contains a cycle $(d_1, d_2, \cdots, d_k)$ of τ, then $\rho\tau\rho^{-1}$ contains the cycle $(\rho(d_1), \rho(d_2), \cdots, \rho(d_k))$. Thus, show that if two permutations in S_n are conjugate, then they have the same cycle type.

Problem 9. [2] Show that the permutations $(123)(45) \in S_9$ and $(567)(89) \in S_9$ are conjugate.

Problem 10. [3] Show that if two permutations in S_n have the same cycle type, then they are conjugate. (Hint: think about the previous two problems.)

3 Signs and Reduced Words (12 points)

We say a transposition is **simple** if it swaps adjacent elements. For example, (12) is simple, but (13) is not.

Fix $1 \leq i < j \leq n$. We say (i, j) is an **inversion** of π if $\pi(i) > \pi(j)$. We let $I(\pi)$ denote the **number** of inversions of π.

Example 1. *In the permutation $\pi = 4213 \in S_4$, $(1, 2)$ is an inversion of π, but $(2, 4)$ is not since $\pi(2) = 2 < 3 = \pi(4)$.*

Note that inversions of a permutation should not be confused with the inverse permutation.

Problem 11. [2] Compute the number of inversions of the transposition (ij) where $i < j$. Express your answer as a function of i, j.

Problem 12. [2] Compute the number of inversions of $(146)(235)$.

We say π is odd if it can be obtained using an odd number of transpositions. Otherwise, we say π is even. This is called the sign of a permutation, usually denoted as sgn (π).

Problem 13. [5] Show that $I(\pi)$ is even if and only if π is even. That is, show that $I(\pi)$ is even if π is even, and that π is even if $I(\pi)$ is even.

In particular, this shows that if a permutation can be attained from the identity using an odd number of transpositions, then it cannot be attained from the identity permutation by using an even number of transpositions.
We define a **word** of a permutation w to be a sequence $(s_1, \cdots, s_k)$ of *simple* transpositions such that $w = s_1 \circ s_2 \circ \cdots \circ s_k$. We define the **length** of a permutation to be the minimum k such that it has a word of length k. The length of the word is denoted $\ell(w)$.

Problem 14. [3] Prove that $I(w) = \ell(w)$ by using the following outline:

i) Show that $\ell(w) \geq I(w)$ by showing that each simple transposition changes the number of inversions by $+1$ or -1.

ii) Show that equality can be attained by inducting on $I(w)$.

4 Applications to the Cube (12 points)

We now explore how permutations can be used to explore the symmetries of the cube with vertices $(\pm 1, \pm 1, \pm 1)$. We define a rotation of a cube to be any multiple of $90°$ rotations of vertices either clockwise or counterclockwise about the x, y, or z axis. We similarly define a reflection of a cube to be a reflection of the vertices about either the xy, yz, or xz plane.

Consider labeling of diagonals of the cube in the following manner:

- $1 \to$ diagonal connecting the points $(1, 1, 1)$ and $(-1, -1, -1)$.
- $2 \to$ diagonal connecting the points $(1, -1, -1)$ and $(-1, 1, 1)$.
- $3 \to$ diagonal connecting the points $(-1, -1, 1)$ and $(1, 1, -1)$.
- $4 \to$ diagonal connecting the points $(1, -1, 1)$ and $(-1, 1, -1)$.

We define a permutation of these diagonals to be the rotational action that takes the vertices of one diagonal to the vertices of a image diagonal. For example, consider the permutation (12)(34). We can verify that a $180°$ counterclockwise rotation about x axis maps the vertices of diagonal 1 to vertices on diagonal 2, and so on for the rest of the diagonals.

Problem 15. [4] We explore how to generate permutations of various cycle types. By generate, we mean to show that there exists a series of rotations and reflections that when applied sequentially, result a given target permutation.

i) Show a method to generate any permutation of cycle type $[0, 2]$ ($e.g.$(12)(34)).

ii) Show a method to generate any permutation of cycle type $[1, 0, 1]$ ($e.g.$(123)(4)).

iii) Show a method to generate any permutation of cycle type $[0, 0, 0, 1]$ ($e.g.$(1234)).

Note that this implies any permutation of 4 numbers can be thought of as a series of reflections and rotations (i.e. a symmetry). We will formally prove this in the following problem.

We now leave our discussion of diagonals and consider the symmetric group S_4 again. In the following context, define actions on permutations a, b to be the composition $a \circ b$, and the action on symmetries is the sequential application of the corresponding rotations.

Problem 16. [8] Given an initial ordering of the vertices, we define symmetries of a cube to be all possible permutations of the vertices you can generate via rotations or reflections of the cube.

i) Show that there are 24 symmetries of a cube.

ii) Show that there exists a mapping between the symmetries of a cube and S_4 that has the property that $\phi(a \circ b) = \phi(a) \circ \phi(b)$, where a, b are symmetries of a cube and $\phi(a), \phi(b)$ are elements of S_4. This mapping should also be bijective (one-to-one) and hold for any elements a, b.

iii) Show that a similar mapping exists between the set of permutations generated by (532)(746) and (1357)(2468) and S_4.