DMM Power Round 2024 Solutions

October 2024

The point values will be determined once it has been testsolved.
For questions asking you to **find**, **evaluate**, **give**, or **compute**, you do not need to give any additional justification. There are no partial credits available for wrong solutions. For questions asking you to **show** or to **prove**, in order to receive full credits you should show a concrete, precise proof. Partial credits are available for these questions.

1 The Symmetric Group

In what follows, let n be a positive integer, and $[n] := \{1, \cdots, n\}$

Definition 1. *We define a* ***permutation*** *of* $[n]$ *to be a bijection* $[n] \to [n]$*. That is, it takes in a number in* $[n]$ *and outputs a unique number in* $[n]$*. Let* S_n *denote the set of permutations of* $[n]$*. It is called* ***the symmetric group on*** n ***elements****.*

There are two ways to think about a permutation:

- We can focus on the values: for example $4213 \in S_4$ denotes the permutation $\pi\colon 1 \mapsto 4, 2 \mapsto 2, 3 \mapsto 1, 4 \mapsto 3$.
- OR we can focus on the cycles: we can consider an arrow $1 \to 2, 2 \to 1$ and $3 \to 3$. We realize that if we start at 1, we return to 1 in two steps, while if we start at 3 we return to it in one step. Thus we can view it as a swap of 1 and 2, so we write this as $(12)(3)$ or as just (12) since we leave out cycles of length 1 by convention. Note that (3) means 3 is mapped to 3.

  Note that this cycle notation is not necessarily unique. For instance, the permutation $4213 \in S_4$ can be written as (143), (431), or (314).

We can multiply permutations by composing them. More specifically, let π, ρ be permutations of $[n]$. Then their product $\pi \circ \rho$ is a permutation satisfying $\pi \circ \rho(j) = \pi(\rho(j))$ for all $1 \le j \le n$. For example, $(345) \in S_5$ is 12453, and $(123) \circ (345) = 23451$.
We use id to denote the special **identity** permutation, which is the permutation that maps $i \mapsto i$ for every i. This has the property that $id \circ \pi = \pi \circ id = \pi$ for any permutation π.

Problem 1. *Evaluate* $(12) \circ (23)$ *and* $(23) \circ (12)$ *in* S_3*.*

Answer: $(12) \circ (23) = 231, \neq (23) \circ (12) = 312$**.**
Next, there is a notion of inverting a permutation, which reverses the effect of the permutation.

Definition 2. *For every permutation* $\pi \in S_n$*, there is a unique permutation* $\rho \in S_n$ *such that* $\rho \circ \pi = id$*. We call* ρ *the* ***inverse*** *of* π*, and we denote it as* $\rho = \pi^{-1}$*.*

Problem 2. *Compute the inverse of the permutation* 35421*.*

Answer: 54132

Problem 3. *Suppose that we have a cycle permutation* $\pi = (c_1 c_2 \cdots c_k)$ *of* $[n]$ *where* $c_i \in [n]$ *for each* $i = 1, \dots, k$*. Compute* π^{-1} *in terms of* c_1 *through* c_k*.*

Answer: $\pi^{-1} = (c_k c_{k-1} \cdots c_1)$

Problem 4. *Show that if $\rho, \pi \in S_n$ satisfies $\rho \circ \pi = id$, then $\pi \circ \rho = id$.*

Definition 3. *We say a* ***transposition*** *is a permutation that swaps two elements and does not change the other elements. For example, (12) is a transposition since it swaps 1 and 2 while preserving every other number. Similarly, (23) and (13) are also transpositions.*

Problem 5. *Give the following permutations in "value" form: for example, instead of writing $(12)(3)$ or (12), write* 213.

- (24)
- $(14) \circ (24)$
- (14)
- *The permutation attained by first swapping the number in* ***positions*** $1, 4$, *then swapping the numbers in* ***positions*** $2, 4$.

Answers:

- 1432
- 4132
- 4231
- 4132

Problem 6. *Fix $k, n \in \mathbb{N}$. Let $s_1, \cdots, s_k$ be* ***transpositions*** *of S_n. We can multiply them $s_1 s_2 \cdots s_k = s_1 \circ s_2 \circ \cdots \circ s_k$. Let $s_i = (a_i b_i)$. Show that $s_1 s_2 \cdots s_k$ will be attained by the following procedure:*

- *Start with the identity permutation.*
- *Swap numbers in positions a_1, b_1*
- *then swap numbers in positions a_2, b_2*
- $\cdots$
- *eventually swap numbers in positions a_k, b_k*

(Hint: Induct on k; start by understanding $k = 1, 2$ well, then show how to go from k to $k+1$. The previous problem should give you a solid understanding of what happens when $k = 2$)

Solution: Induct on k.
The base case, $k = 1$, holds because $(a_1 b_1) = (a_1 b_1)$
Inductive step: We use the problem for $k - 1$ to show the problem for k: by inductive hypothesis on $k - 1$, when I swap numbers in positions a_1, b_1, then numbers in positions a_2, b_2, then so on, so on, I get the permutation $s_1 \circ \cdots \circ s_{k-1}$. I want to show if I swap the numbers in positions a_k, b_k of the permutation $\pi := s_1 \circ \cdots \circ s_{k-1}$, I get the permutation $\pi \circ (a_k b_k)$.
We check three scenarios:
If $c \notin \{a_k, b_k\}$ then $(\pi \circ (a_k b_k))(c) = \pi(c)$, which is what I get by starting with π and swapping numbers in positions a_k, b_k.
We check the two permutations agree on a_k. $(\pi \circ (a_k b_k))(a_k) = \pi(b_k)$, which is what I get by first applying π and swapping the numbers in positions a_k, b_k.
Analogously, the two permutations agree on b_k.

2 Cycle Type

Definition 4. *We say the **cycle type** of a permutation $\pi \in S_n$ to be a sequence $(c_i)_{i\geq 1}$, where c_i is the number of cycles of length i.*

Note that $(c_i)_{i\geq 1}$ is eventually zero. We usually omit the eventually zero terms.
For example, the cycle type of the permutation $23154 \in S_5$ is $[0, 1, 1]$. We would write this permutation as $(123)(45)$. (Note: here, when we write it in cycle form, the cycles are disjoint)

Problem 7. *Compute the number of permutations of S_5 with cycle type $[0, 1, 1]$.*

Solution: There are $\binom{5}{3} = 10$ choices for which three numbers are placed in the cycle of length 3. There are 2 ways to orient the cycle of length 3, so there are a total of 20 permutations in S_5 with cycle type $[0, 1, 1]$.
Sometimes, for $d_1, d_2, \cdots, d_m \in [n]$, we write $(d_1, \cdots, d_m)$ for the cycle that takes d_i to d_{i+1} (where $d_{m+1} = d_1$). For example, (123) may be written as $(1, 2, 3)$. The commas are there to avoid confusion with multiplication.

Definition 5. *We say two permutations $\sigma, \tau \in S_n$ are **conjugate** if there exists $\rho \in S_n$ such that $\sigma = \rho\tau\rho^{-1}$.*

Problem 8. *Show that a cycle $(d_1, d_2, \cdots, d_k)$ of τ corresponds to a cycle $(\rho(d_1), \rho(d_2), \cdots, \rho(d_k))$ of $\rho\tau\rho^{-1}$. Thus, show that if two permutations in S_n are conjugate, then they have the same cycle type.*

Direct computation. $(\rho\tau\rho^{-1})(\rho(d_i)) = (\rho\tau)(d_i) = \rho(d_{i+1})$.

Problem 9. *Show that the permutations $(123)(45) \in S_9$ and $(567)(89) \in S_9$ are conjugate.*

Let $\tau = (123)(45)$. There are many possible choices of ρ such that $\rho\tau\rho^{-1} = (567)(89)$. Here's one: ρ: $567891234 \in S_9$. $\rho = 756984213$ also works.

Problem 10. *Show that if two permutations in S_n have the same cycle type, then they are conjugate. (Hint: think about the previous two problems.)*

Call the two permutations π, σ. We construct a size-preserving bijection between the cycles, meaning a cycle of length l in π is taken to a cycle of length l in σ. Say a cycle $(d_1, d_2, \cdots, d_k)$ in π is mapped to a cycle $(e_1, e_2, \cdots, e_k)$ in σ, then we set $\rho(d_i) = e_i$. Note ρ is a bijection of $[n]$ because the cycles I choose are disjoint, and ρ is clearly injective.

3 Signs and Reduced Words

We say a transposition is **simple** if it swaps adjacent elements.
For example, (12) is simple, (13) is not.
Fix $1 \leq i < j \leq n$. We say (i, j) is an **inversion** of π if $\pi(i) > \pi(j)$. We let $I(\pi)$ denote the **number** of inversions of π.

Example 1. *In the permutation $\pi = 4213 \in S_4$, $(1, 2)$ is an inversion of π, but $(2, 4)$ is not since $\pi(2) = 2 < 3 = \pi(4)$.*

Note that inversions of a permutation should not be confused with the inverse permutation.

Problem 11. *Compute the number of inversions of the transposition (ij) where $i < j$. Express your answer as a function of i, j.*

The inversions are $\{(i,l)\colon i+1 \le l \le j-1\} \cup \{(l,j)\colon i+1 \le l \le j-1\} \cup \{(i,j)\}$**. Hence there are** $2(j-i)-1$ **inversions.**

Problem 12. *Compute the number of inversions of* $(146)(235)$.

The permutation is $435621 \in S_6$**. The inversions are** $(1,2),(1,5),(1,6),(2,5),(2,6),(3,5),(3,6),(4,5),(4,6),(5,6)$**. There are** 10 **inversions.**

We say π is odd if it can be obtained using an odd number of transpositions. Otherwise, we say π is even. This is called the sign of a permutation, usually denoted as sgn (π).

Problem 13. *Show that* $I(\pi)$ *is even if and only if* π *is even.*

Solution: It suffices to show that if π **is an arbitrary permutation (say in** S_n**) and** $(i,i+1)$ **is a transposition, then** $I((i,i+1)\circ\pi) - I(\pi)$ **is odd. This implies the problem, since the transpositions clearly generate the symmetric group, and a transposition** (12) **is odd and** $I((12)) = 1$**.**

Suppose $i<j$**,** $\pi(a)=i$ **and** $\pi(b)=j$**. Then** $((i,j)\circ\pi)(a)=j$ **and** $((i,j)\circ\pi)(b)=i$**. We can WLOG** $a<b$ **as well, for the other case can be handled similarly. We can verify that**

$$I((i,j)\circ\pi) - I(\pi) = 1 + 2\#(x\colon i<\pi(x)<j, a<x<b),$$

because (i,j) **is an inversion in** $(i,j)\circ\pi$ **but not in** π**, and for** $a<x<b$**,** $(a,x),(x,b)$ **is an inversion in** $(i,j)\circ\pi$ **but not in** π **iff** $i<\pi(x)<j$**.**

We define a **word** of a permutation π to be a sequence $(s_1,\cdots,s_k)$ of *simple* transpositions such that $\pi = s_1 s_2 \cdots s_k$. We say the **length** of a permutation is the minimum k such that there exists such a sequence. The length of the word is denoted $\ell(\pi)$.

Problem 14. *Prove that* $I(\pi)=\ell(\pi)$ *by using the following outline:*

- *Show that* $\ell(\pi) \ge I(\pi)$ *by showing that each simple transposition changes the number of inversions by* $+1$ *or* -1.

  Define a,b ***such that*** $\pi(a)=i, \pi(b)=i+1$ ***(better:*** $a=\pi^{-1}(i), b=\pi^{-1}(i+1)$***). If*** $a<b$***, then***

  $$I((i,i+1)\circ\pi) - I(\pi) = 1 + 2\#(x\colon i<\pi(x)<i+1, a<x<b) = 1$$

  Thus, it's not hard to check that

  $$I((i,i+1)\circ\pi) - I(\pi) = \begin{cases} 1 & \text{if } a<b \\ -1 & \text{if } a>b \end{cases}$$

  This implies that $l(s_k \circ s_{k-1} \circ \cdots \circ s_1) \le k$ ***for all*** k***.***

- *Show that equality can be attained by inducting on* $I(w)$.

  For the inductive step, simply find j ***such that*** $\pi^{-1}(j) > \pi^{-1}(j+1)$ ***and apply inductive hypothesis on the permutation*** $(j,j+1)\circ\pi$***, which exists unless*** $\pi = id$

4 Applications to the Cube (12 points)

We now explore how permutations can be used to explore the symmetries of the cube with vertices $(\pm1,\pm1,\pm1)$. We define a rotation of a cube to be any multiple of $90°$ rotations of vertices either clockwise or counterclockwise about the x, y, or z axis. We similarly define a reflection of a cube to be a reflection of the vertices about either the xy,

yz, or xz plane.

Consider labeling of diagonals of the cube in the following manner:

- $1 \to$ diagonal connecting the points $(1, 1, 1)$ and $(-1, -1, -1)$.
- $2 \to$ diagonal connecting the points $(1, -1, -1)$ and $(-1, 1, 1)$.
- $3 \to$ diagonal connecting the points $(-1, -1, 1)$ and $(1, 1, -1)$.
- $4 \to$ diagonal connecting the points $(1, -1, 1)$ and $(-1, 1, -1)$.

We define a permutation of these diagonals to be the rotational action that takes the vertices of one diagonal to the vertices of a image diagonal. For example, consider the permutation (12)(34). We can verify that a 180° counter-clockwise rotation about x axis maps the vertices of diagonal 1 to vertices on diagonal 2, and so on for the rest of the diagonals.

Problem 15. [4] We explore how to generate permutations of various cycle types. By generate, we mean to show that there exists a series of rotations and reflections that when applied sequentially, result a given target permutation.

1. Show a method to generate any permutation of cycle type $[0, 2]$ (*e.g.*(12)(34)).
2. Show a method to generate any permutation of cycle type $[1, 0, 1]$ (*e.g.*(123)(4)).
3. Show a method to generate any permutation of cycle type $[0, 0, 0, 1]$ (*e.g.*(1234)).

Note that this implies any permutation of 4 numbers can be thought of as a series of reflections and rotations (i.e. a symmetry). We will formally prove this in the following problem.

Solution.

1.

- $(12)(34) \to$ rotation 180° of front face of cube
- $(13)(24) \to$ rotation 180° of top face of cube
- $(14)(23) \to$ rotation 180° of right face of cube

2.

- $(123) \to$ rotation 120° with diagonal 4 as axis
- $(132) \to$ rotation 240° with diagonal 4 as axis
- $(124) \to$ rotation 120° with diagonal 3 as axis
- $(142) \to$ rotation 240° with diagonal 3 as axis
- $(134) \to$ rotation 120° with diagonal 2 as axis
- $(143) \to$ rotation 240° with diagonal 2 as axis
- $(234) \to$ rotation 120° with diagonal 1 as axis
- $(243) \to$ rotation 240° with diagonal 1 as axis

3.

- $(1234) \to$ rotation 90° around axis $(0, 0, 1), (0, 0, -1)$
- $(1432) \to$ rotation 270° around axis $(0, 0, 1), (0, 0, -1)$
- $(1324) \to$ rotation 90° around axis $(0, 1, 0), (0, -1, 0)$

- $(1423) \to$ rotation $270°$ around axis $(0,1,0),(0,-1,0)$
- $(1243) \to$ rotation $90°$ around axis $(1,0,0),(-1,0,0)$
- $(1342) \to$ rotation $270°$ around axis $(1,0,0),(-1,0,0)$

We now leave our discussion of diagonals and consider the symmetric group S_4 again. In the following context, define actions on permutations a, b to be the composition $a \circ b$, and the action on symmetries is the sequential application of the corresponding rotations.

Problem 16. [8] Given an initial ordering of the vertices, we define symmetries of a cube to be all possible permutations of the vertices you can generate via rotations or reflections of the cube.

1. Show that there are 24 symmetries of a cube.
2. Show that there exists a mapping between the symmetries of a cube and S_4 that has the property that $\phi(a \circ b) = \phi(a) \circ \phi(b)$, where a, b are symmetries of a cube and $\phi(a), \phi(b)$ are elements of S_4. This mapping should also be bijective (one-to-one) and hold for any elements a, b.
3. Show that a similar mapping exists between the set of permutations generated by $(532)(746)$ and $(1357)(2468)$ and S_4.

Solution.

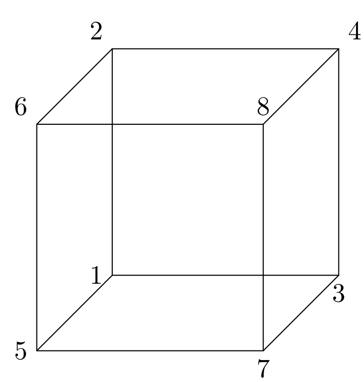
1. Label the vertices of the cube so that the bottom face is labeled $1-4$ and the top face is labeled $5-8$, with edges of the cube connecting $1-5$, $2-6$, etc. Note that there are 6 choices for the bottom face and then 4 ways to rotate the cube, this results in 24 permutations of the vertices, and thus there are 24 symmetries of the cube.

2. We already have shown that we can get 17 of the permutations in Problem 15. We are missing the identity symmetry (which maps to the identity permutation), and the pairwise permutations of $(12),(13),(14),(23),(24),(34)$. We can generate them in the following manner:

- $(12) = (1234)(243)$
- $(13) = (1324)(342)$
- $(14) = (1432)(423)$
- $(23) = (2314)(341)$
- $(24) = (2413)(431)$
- $(34) = (3412)(421)$

The identity element of S_4 is simply the self-preserving 'no rotation' of the cube. We have now shown that all of S_4 can be represented as symmetries of the cube. Since permutations uphold the property of $\phi(a \circ b) = \phi(a) \circ \phi(b)$, we have that the mapping with symmetries of cube also uphold this property. Since we have shown that each of them can be generated via an element of S_24, and we know that both sets have size 24, this mapping must be bijective.

3. Consider the following labeling of the vertices:



Now note that the permutation (532)(746) is a 90° rotation of the cube about diagonal 4, and the permutation (1357)(2468) is a rotation of the cube 90° about the x-axis. Following the generations from the previous two problems, we can see that these two symmetries generate all symmetries of the cube. So, by part 2, a similar mapping exists.