2024 DUKE MATH MEET RELAY ROUND SOLUTIONS

1 Relay Round - Set 1

1. Each interior angle of a regular pentagon is 108° and 60° for an equilateral triangle. Thus, $\angle FBC = 48^\circ$. Because $BF = BC$, $\triangle FBC$ is isosceles, so $\angle BFC = \angle BCF = 66^\circ$. We then have $\angle FCD = 108^\circ - \angle BCF = \boxed{42^\circ}$.

2. From Problem 1 we have $T = 42$. We take the digits of T (i.e. $\{4, 2\}$) and add one extra digit $d \in \{0, 1, 2, 3, 4, 5\}$. We must form a 3-digit number N that is divisible by 4 (so its last two digits form a multiple of 4) and $N > 100$.

   We do a short casework on d.

   - $d = 0$: digits $\{0, 2, 4\}$. Valid endings (multiples of 4) are $20, 40, 24, 04$. Only $420, 240, 204$ are three-digit: 3 numbers.
   - $d = 1$: digits $\{1, 2, 4\}$. Valid endings: $12, 24 \Rightarrow$ numbers $412, 124$: 2.
   - $d = 2$: digits $\{2, 2, 4\}$. Valid ending: $24 \Rightarrow$ number 224: 1.
   - $d = 3$: digits $\{2, 3, 4\}$. Valid endings: $24, 32 \Rightarrow$ numbers $324, 432$: 2.
   - $d = 4$: digits $\{2, 4, 4\}$. Valid endings: $24, 44 \Rightarrow$ numbers $424, 244$: 2.
   - $d = 5$: digits $\{2, 4, 5\}$. Valid endings: $24, 52 \Rightarrow$ numbers $524, 452$: 2.

   Total: $3 + 2 + 1 + 2 + 2 + 2 = \boxed{12}$.

3. Let $AB = x$ and $AE = y$. Since $\triangle CFD \sim \triangle DEB$, we have

$$\frac{2-y}{y} = \frac{y}{x-y} \quad \Longrightarrow \quad y = \frac{2x}{x+2}.$$

   Then

$$HC = HF - CF = y - (2-y) = 2y - 2 = \frac{2x-4}{x+2}, \qquad HG = y = \frac{2x}{x+2}.$$

   Apply the Pythagorean Theorem to right triangle HCG (note $CG = 2$):

$$HC^2 + HG^2 = \frac{4(x-2)^2}{(x+2)^2} + \frac{4x^2}{(x+2)^2} = 4 \cdot \frac{(x-2)^2 + x^2}{(x+2)^2} = 4.$$

   Thus

$$(x-2)^2 + x^2 = (x+2)^2 \implies 2x^2 - 4x + 4 = x^2 + 4x + 4 \implies x^2 - 8x = 0 \implies x = 8.$$

   Finally,

$$[ABC] = \frac{AC \cdot AB}{2} = \frac{2 \cdot 8}{2} = \boxed{8}.$$

2 Relay Round - Set 2

1. Since $64 = 2^6$, every $1 \leq n \leq 63$ has a 6-bit binary form. We can count based on the number of 1's:
$$\text{one } 1 = \binom{6}{1} = 6, \qquad \text{two } 1 = \binom{6}{2} = 15.$$
Thus the total is $6 + 15 = \boxed{21}$. Hence $T = 21$.

2. Solve for $x^3 + \frac{1}{x^3}$ given $x^2 + \frac{1}{x^2} = \frac{T}{3} = \frac{21}{3} = 7$. Let $t = x + \frac{1}{x}$. Then $t^2 - 2 = 7 \Rightarrow t^2 = 9 \Rightarrow t = \pm 3$. Now
$$x^3 + \frac{1}{x^3} = t^3 - 3t = \begin{cases} 3^3 - 3 \cdot 3 = 18, \\ (-3)^3 - 3(-3) = -18. \end{cases}$$
The product of all possible values is $18 \cdot (-18) = \boxed{-324}$.

3. With two fair dice,
$$p_4 = \Pr(\text{sum } 4) = \frac{3}{36} = \frac{1}{12}, \qquad p_5 = \Pr(\text{sum } 5) = \frac{4}{36} = \frac{1}{9}.$$
Let P be the probability that A eventually wins. One "cycle" is A then B; if both miss, we return to the same state. Hence
$$P = p_4 + (1 - p_4)(1 - p_5)P \quad \Longrightarrow \quad P = \frac{p_4}{1 - (1 - p_4)(1 - p_5)} = \frac{p_4}{p_4 + p_5 - p_4 p_5}.$$
Substitute the values:
$$P = \frac{\frac{1}{12}}{\frac{1}{12} + \frac{1}{9} - \frac{1}{108}} = \frac{\frac{1}{12}}{\frac{5}{27}} = \frac{27}{60} = \boxed{\frac{9}{20}}.$$