2024 DUKE MATH MEET TEAM ROUND

1. What is the smallest positive integer k such that $2 \cdot k$ is a perfect square and $3 \cdot k$ is a perfect cube?

2. Nikhil loves baking cookies. He flattens out a circle of dough with radius 20. One of the cookie cutters is a circle of radius 5 and is laid internally tangent to the circle of the dough. Another is a square with length x. The square cookie can be cut out tangent to the circle cutter and has two corners on the edge of the dough. The length x of the cookie cutter can be expressed $a + \sqrt{b}$, for positive integers a, b. Find $a + b$.

3. Determine the number of pairs of 3-digit palindromic numbers (x, y) such that the sum $x + y$ is a 4-digit palindromic number. A palindromic number is defined as a number that remains the same when its digits are reversed. For example, $x = 232$ and $y = 989$ yield $x + y = 1221$, which is a 4-digit palindrome.

4. Given $x^2 - x - 1 = 0$, compute
$$\frac{x^{16} - 1}{x^8 + 2x^7}$$

5. An ant begins at $(1, 0)$ in the coordinate plane, facing east. At the kth second (starting with $k = 1$), the ant decides to turn counterclockwise either $60°$ or $300°$ with equal probability, and then moves $1/2^k$ units forward. It is given that this process always converges to some point. Then, let a be the expected x-coordinate of where it converges. What is $3a$?

6. The sum of the positive roots of $(x^2 - 3x - 4)^2 - 3(x^2 - 3x - 4) - 4 - x = 0$ can be expressed as $a + b\sqrt{c} + \sqrt{d}$, where c and d are square-free and $c \neq d$. Find $a + b + c + d$.

7. Triangle ABC has side lengths $AB = 16$, $BC = 19$, and $CA = 21$. Square $DUKE$ is in the interior of ABC such that: U is on $\overline{AB}$, K is on $\overline{BC}$, and E is on $\overline{CA}$. Given that $\angle AUD = \angle AED$, find the ratio of lengths $AD : DK$. Note: the vertices of ABC and $DUKE$ are given in counterclockwise order. The answer can be expressed as $\frac{\sqrt{a}-b}{c}$, where a is square-free. Find $a + b + c$.

8. Define $\nu_2(n)$ to be the exponent of the maximum power of 2 that divides n. For an integer n, let $C(n) = \binom{2n}{n}$. As n ranges from 1 to 2024, what is the maximum value of $\nu_2(C(n))$?

9. Let integers a and b, $a \neq b$ be the solutions to
$$(a^2 - 6(b - a))(a^2 + 9(b - a)) = ab^3$$
If (a_i, b_i) represent all such solution pairs, find the sum of all b_i.

10. Circles w_1 and w_2 meet at P, Q, and let P lie on segment AB, where A is a point on w_1 and B is a point on w_2. Let D be on the minor arc PQ, and let C be the intersection of PD with circle w_1 that is not P. Ray $\overline{BD}$ and $\overline{AC}$ meet at X. Point Y is on w_1 such that $\overline{PY} || \overline{BD}$. Point Z is on w_2 such that $\overline{PZ} || \overline{AC}$. Given $\angle PYX = 20°$ and $\angle CYZ = 30°$, find $\angle PAC + \angle QBD$.