2024 DUKE MATH MEET TEAM ROUND SOLUTIONS

1. Note that perfect squares have all even exponents and perfect cubes have exponents all divisible by 3. It is clear that any prime factors of k other than $2, 3$ are unnecessary, so we have $k = 2^a \cdot 3^b$, where $a \equiv 0 \pmod 3$, $a + 1 \equiv 0 \pmod 2$, $b \equiv 0 \pmod 2$, $b + 1 \equiv 0 \pmod 3$. From these equations, we see the smallest possible $k = 2^3 \cdot 3^2 = \boxed{72}$.

2. Call the center of the larger circle O. Extend the diameter $\overline{PQ}$ to the other side of the square (at point E), and draw $\overline{AO}$. We now have a right triangle with hypotenuse 20. Since $OQ = OP - PQ = 20 - 10 = 10$, we know that $OE = AB - OQ = AB - 10$. The other leg, AE, is just $\frac{1}{2}AB$. Apply the Pythagorean Theorem:

$$(AB - 10)^2 + \left(\tfrac{1}{2}AB\right)^2 = 20^2.$$

Expanding and simplifying,

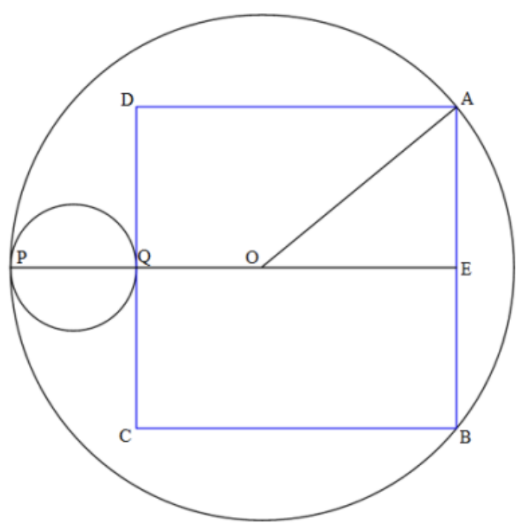
$$AB^2 - 20AB + 100 + \tfrac{1}{4}AB^2 - 400 = 0 \quad \Rightarrow \quad AB^2 - 16AB - 240 = 0.$$

By the quadratic formula,

$$AB = \frac{16 \pm \sqrt{16^2 + 4 \cdot 240}}{2} = 8 \pm \sqrt{304}.$$

Discard the negative root, so the answer is

$$\boxed{8 + \sqrt{304}}.$$



3. Let the two 3-digit palindromes be

$$x = aba, \qquad y = cdc,$$

so their sum is a 4-digit palindrome

$$\begin{array}{r} a\ b\ a \\ +\ c\ d\ c \\ \hline e\ f\ f\ e \end{array}$$

with leading digit $e = 1$ (the sum must start with 1).

(a) From the units column, $a + c = 1$ or $a + c = 11$. The carry into the thousands place forces $a + c = 11$.

(b) Hence the possible pairs (a, c) (up to order) are

$$(2, 9),\ (3, 8),\ (4, 7),\ (5, 6).$$

(c) From the tens column we have $b + d + 1 = f$ (if no carry to the hundreds place) or $b + d + 1 = f + 10$ (if there is a carry to the hundreds place).

- *Case 1 (no carry to hundreds):* $f = 1$ and $b + d = 0$, so $b = d = 0$. This gives 4 solutions:

$$202 + 909 = 1111, \quad 303 + 808 = 1111, \quad 404 + 707 = 1111, \quad 505 + 606 = 1111.$$

- *Case 2 (carry to hundreds):* $f = 2$ and $b + d = 11$. The ordered digit pairs (b, d) with $b, d \in \{0, \dots, 9\}$ and $b + d = 11$ are

$$(2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3), (9, 2),$$

so there are 8 choices for (b, d) for each of the 4 choices of (a, c), yielding $4 \cdot 8 = 32$ solutions.

(d) Total solutions: $4 + 32 = 36$.

Therefore, the number of ordered pairs (x, y) of 3-digit palindromes with $x+y$ also palindromic is

$$2 \times 36 = \boxed{72}.$$

4. From $x^2 - x - 1 = 0$ we have $x^2 = x + 1$, hence

$$x^{n+1} = x \cdot x^n = x^{n-1}(x^2) = x^{n-1}(x + 1) = x^n + x^{n-1}.$$

By induction this gives

$$x^n = F_n x + F_{n-1} \quad (n \geq 1),$$

where (F_n) are Fibonacci numbers $(F_0 = 0, F_1 = 1)$. Thus

$$x^7 = 13x + 8, \qquad x^8 = 21x + 13, \qquad x^{16} = 987x + 610.$$

Therefore

$$\frac{x^{16} - 1}{x^8 + 2x^7} = \frac{987x + 610 - 1}{(21x + 13) + 2(13x + 8)} = \frac{987x + 609}{47x + 29} = \boxed{21}.$$

5. Let X_i be the random variable with

$$\Pr(X_i = \tfrac{\pi}{3}) = \Pr(X_i = -\tfrac{\pi}{3}) = \tfrac{1}{2}, \qquad i = 1, 2, \ldots$$

(these indicate the ant's turn at the ith second, in radians). The point P to which the process converges can be written

$$P \;=\; 1 + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k e^{\,i(X_1+\cdots+X_k)}.$$

Since the X_i are independent,

$$\mathbb{E}\big[e^{\,i(X_1+\cdots+X_k)}\big] = \prod_{j=1}^{k} \mathbb{E}\big[e^{\,iX_j}\big] = \Big(\mathbb{E}[e^{\,iX_1}]\Big)^k.$$

Now

$$\mathbb{E}[e^{\,iX_1}] = \tfrac{1}{2}\big(e^{i\pi/3} + e^{-i\pi/3}\big) = \cos\big(\tfrac{\pi}{3}\big) = \tfrac{1}{2}.$$

By linearity of expectation,

$$\mathbb{E}[P] = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^k = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = 1 + \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{4}{3}.$$

Therefore, the expected point of convergence is $\big(\tfrac{4}{3}, 0\big)$, and so

$$3 \cdot \tfrac{4}{3} + 0 = \boxed{4}.$$

6. We notice that the equation $(x^2-3x-4)^2-3(x^2-3x-4)-4-x=0$ resembles the composition of function $f(x) = x^2-3x-4$ with itself. Indeed, $f(f(x)) = (x^2-3x-4)^2-3(x^2-3x-4)-4$ and by moving the x term in our original equation, we find that $f(f(x)) = x$.

   The equation $f(f(x)) = x$ always has the solution $f(x) = x$, as substituting $f(x) = x$ into $f(f(x)) = x$ yields $f(x) = x$ which is true. From that we get that $x^2 - 3x - 4 = x$, or $x^2 - 4x - 4 = 0$ is a solution to our equation.

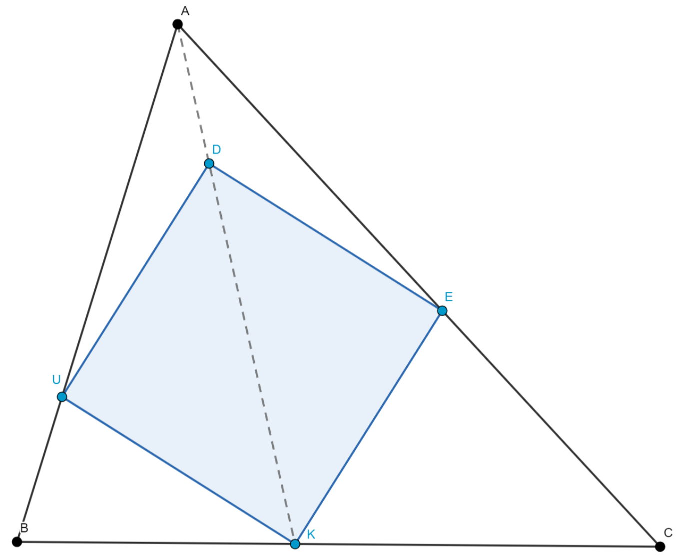
   Now, we know that because $(x^2-3x-4)^2-3(x^2-3x-4)-4-x$ is a monic fourth degree polynomial (by looking at $(x^2)^2$, we know that it equals the product of (x^2-4x-4) and some other factor (x^2+ax+b), yielding

$$(x^2-4x-4)(x^2+ax+b) = (x^2-3x-4)^2-3(x^2-3x-4)-4-x = x^4+cx^3+dx^2+ex+f$$

   for some coefficients c, d, e, and f.

   The finish is simple. Equating coefficients, we know $(-4)\cdot b = f = (-4)^2 - 3(-4) - 4 = 24 \longrightarrow b = -6$. Likewise, $(-4)a + (-4)b = e = 2(-3)(-4) - 3(-3) - 1 = 32 \longrightarrow a = -2$. All that remains is to find the sum of the positive roots of the two quadratics x^2-4x-4 and x^2-2x-6, which is $(2+2\sqrt{2}) + (1+\sqrt{7}) = 3 + 2\sqrt{2} + \sqrt{7} \implies \boxed{14}$

7. Reference the diagram below:



The condition $\angle AUD = \angle AED$ gives us symmetry, which coupled with the fact $DU = DE$ allows us to conclude AD is the angle bisector of $\angle BAC$ and that A, D, K are collinear. Furthermore, applying law of cosines we find

$$\cos(\angle BAC) = \frac{16^2 + 21^2 - 19^2}{2 \cdot 16 \cdot 21} = \frac{1}{2},$$

so $\angle BAC = 60°$.

Since we are only interested in the ratio $AD : DK$, we can avoid direct computations of lengths. Instead, we express everything in terms of d, the side length of $DUKE$. Evidently, $DK = d\sqrt{2}$. For the length AD, we have from before that $\angle DAE = 30°$. Also, by collinearity of A, D, K, we have

$$\angle ADE = 180° - \angle KDE = 135°.$$

With these two facts, we obtain $\angle AED = 15°$. Using law of sines, we equate

$$\frac{AD}{\sin(15°)} = \frac{DE}{\sin(30°)} = \frac{d}{\sin(30°)},$$

which allows us to solve

$$AD = d \cdot \frac{\sin(15°)}{\sin(30°)} = d \cdot \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{1}{2}} = \frac{\sqrt{6}-\sqrt{2}}{2}d.$$

Hence, our final answer is

$$\frac{AD}{DK} = \frac{\frac{\sqrt{6}-\sqrt{2}}{2}d}{d\sqrt{2}} = \frac{\sqrt{3}-1}{2}. \implies \boxed{6}$$

8. Note that

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} = \frac{2^n \cdot n![(2n-1)(2n-3)\cdots(3)(1)]}{n!n!} = \frac{2^n[(2n-1)(2n-3)\cdots(3)(1)]}{n!}.$$

The numerator has n powers of 2, while the denominator has $\nu_2(n!)$ powers. So, we want to maximize $n - \nu_2(n!)$. By Legendre's formula, we have

$$\nu_2(n) = \frac{n - S_2(n)}{2-1} = n - S_2(n),$$

where $S_2(n)$ is the sum of digits of n when written in base 2. So, our desired expression is

$$n - (n - S_2(n)) = S_2(n).$$

Thus, we want the maximum sum of binary digits for an integer from 1 to 2024. Note that

$$2^{10} - 1 \leq 2024 < 2^{11} - 1,$$

so the maximum number we can get is $\boxed{10}$.

9. Expanding, and noticing that $a = b$ is a solution the equation, we can factor the eqution as:

$$(a-b)(ab^2 + a^2b + 54b + a^3 - 3a^2 - 54a) = 0$$

Since a and b are distinct integers, we ignore the $(a-b)$ factor and focus on when $(ab^2 + a^2b + 54b + a^3 - 3a^2 - 54a) = 0$.

To solve this, consider it to be a quadratic in b and let us write the auxiliary quadratic equation:

$$a\chi^2 + (a^2 + 54)\chi + a(a+6)(a-9) = 0$$

For this to have integer solutions for $\chi = b$, we know that the discriminant has to be a square integer.

$$\Delta = (a^2 + 54)^2 - 4 * a * a(a+6)(a-9) \longrightarrow \Delta = -3a^4 + 12a^3 + 324a^2 + 2916$$

This is quite helpful, as we know that due to the negative leading coefficient, $a \to \infty$ and $a \to -\infty$ sends Δ to $-\infty$. Thus we can bound a.

To find these bounds, we notice that this is the same as finding the roots of the function $f(a) = -3a^4 + 12a^3 + 324a^2 + 2916$, which is like solving $\frac{f(a)}{3} = g(a) = -a^4 + 4a^3 + 108a^2 + 972 = 0$.

We notice that for smallish values of a, the value of the function is largely dictated by the $-a^4$ and $108a^2$ term. To cancel these out, we set $a = -10$ and notice that $f(-10) < 0$. On the positive side, we find that $a = 10$ is not enough to have $f(10) < 0$ and by simple additional guess-and-check we get that $f(13) < 0$. To be certain these are the bounding values, we rewrite the function $g(a)$ as $g(a) = a^2(-a^2 + 4a + 108) + 972$. Because a^2 will only increase as $|a|$ increases, we can focus on $h(a) = -a^2 + 4a + 108$ and observe that this quadratic is monotonically decreasing from $a : -10 \to -\infty$ and $a : 13 \to \infty$. Thus, we derive the bound $-9 \leq a \leq 12$

Now we need to solve the actual discriminant $3(-a^4 + 4a^3 + 108a^2 + 972) = n^2$ for integers a and n. Since we can factor out a 3 on the left hand side, we know $3 \mid n$ ($n = 3m$ for some $m \in \mathbb{Z}$). Thus, $-a^4 + 4a^3 + 108a^2 + 972$ needs to be divisible by 3 as well. Examining $\pmod 3$ we get $a^3(4 - a) \equiv 0 \pmod 3$, which only holds for $a = 0$ or $a = 1 \pmod 3$. So $a = 3c$ or $a = 3c + 1$ for $-3 \leq c \leq 4$, $c \in \mathbb{Z}$.

Substituting $n = 3m$ and $a = 3c$ we get $9(-3c^4 + 4c^3 + 36c^2 + 36) = m^2$. Disregarding the 9 and checking whether or not $-3c^4 + 4c^3 + 36c^2 + 36$ is a perfect square for $-3 \leq c \leq 4$ gives us the solution pairs $(a, |n|) : (-9, 27)$, $(-6, 90)$, $(0, 54)$, $(9, 135)$, $(12, 90)$.

Substituting $n = 3m$ and $a = 3c + 1$ we get:

$$9(-27c^4 + 342c^2 + 224c + 361) = 9m^2 \longrightarrow -27c^4 + 342c^2 + 224c + 361 = m^2$$

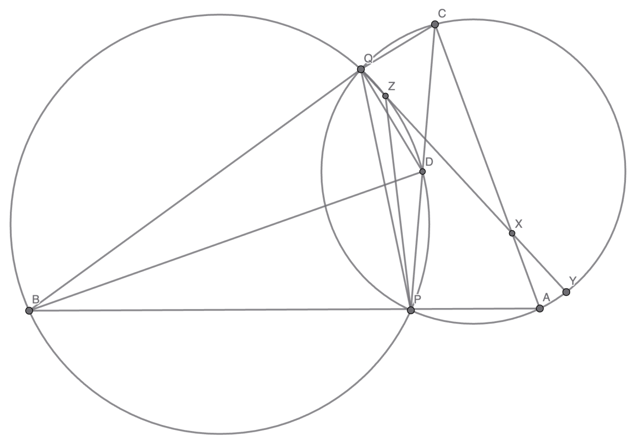
Checking the values $-3 \leq c \leq 3$ (not 4 as $c = 3 \cdot 4 + 1 = 13$, which is beyond our bound) is also pretty simple, giving us the solutions pairs $(a, |n|) : (1, 57)$, $(4, 90)$.

From the quadratic formula solution for the quadratic in b previously, we have:

$$b = \frac{-(a^2 + 54) \pm |n|}{2a}$$

Using this equation and after taking out the unwanted pairs $(1, 1)$ and $(12, -\frac{9}{2})$, we can add all the b_i of solution pairs (a_i, b_i) to get: $6 + 9 + 0 + 15 - 56 - 20 - 15 + 0 - 12 = \boxed{-73}$

10. Consider the diagram below: Since $Q \in w_1 \cap w_2$, Q is the center of the spiral similarity



sending chord $\overline{AC} \subset w_1$ to chord $\overline{BD} \subset w_2$. Equivalently, Q is the Miquel point of the complete quadrilateral formed by the four lines AC, BD, AD, CB. In particular

$$Q, B, A, X \text{ are concyclic} \quad \text{and} \quad Q, C, X, D \text{ are concyclic.}$$

We will write these as $(QBAX)$ and $(QCXD)$.

Let $Y' \in w_1$ be the unique point with $PY' \parallel BX$. Then,

$$\angle QY'P = \angle QCP \quad \text{(same arc } QP \text{ on } w_1\text{)},$$

$$\angle QCP = \angle QCD \quad \text{(since } C, P, D \text{ are collinear)},$$

$$\angle QCD = \angle QXD \quad \text{(equal angles subtending arc } QD \text{ in } (QCXD)\text{)},$$

and, because Q is the spiral center sending CX to XB,

$$\angle QXD = \angle QXB \quad \text{(from similarity } \triangle QXD \sim \triangle QXA \text{ or } \triangle QCX \sim \triangle QXB\text{)}.$$

Hence

$$\angle QY'P = \angle QXB.$$

By construction this says precisely that the angle a line through P parallel to BX makes with QP equals the angle QXB, i.e. $PY' \parallel BX$. But BX and BD are the same line, while Y was the point of w_1 with $PY \parallel BD$; by uniqueness of the parallel through P meeting w_1 we conclude

$$Y' = Y.$$

In particular, all angle identities above hold with Y in place of Y'.

Define $Z' \in w_2$ by $PZ' \parallel AX$. Repeating the same argument with roles of w_1 and w_2 swapped (now using that the spiral at Q sends AX to XD and AC to BD), we obtain

$$\angle QZ'P = \angle QXA.$$

But AX and AC are the same line through A and X, and Z was defined by $PZ \parallel AC$; hence $Z' = Z$ and P, X, Y, Z are all collinear.

This gives us that $\angle PAC = \text{arc}(PQ) + \text{arc}(QC) = \angle PYQ + \angle QYC = 20^\circ + 30^\circ = 50^\circ$. We also get that $\angle QBD = \angle QPD = \angle QPC = \angle QYC = 30^\circ$. Combining these two gives our answer of $\boxed{80^\circ}$.