Tiebreaker Round Solutions

DMM 2024

1 Tiebreaker

Problem 1: Let $f_1(x) = x + 3$, and for $n \geq 1$ define

$$f_{n+1}(x) = (1 + x)f_n(x) + (-1)^n(2x + 4).$$

Then the two real roots of $f_{2024}(x)$ can be expressed as $a \pm b \sqrt[2024]{c}$, for some integers a, b, c with b and c positive. What is $a + b + c$?

Solution. Let $g_n(x) = f_n(x - 1)$. Then, we have that $g_1(x) = x + 2$ and

$$g_{n+1}(x) = xg_n(x) + (-1)^n(2x + 2)$$

for all $n \geq 1$. Computing the first few values of g_n shows that $g_2(x) = x^2 - 2$ and $g_3(x) = x^3 + 2$, so we suspect that $g_n(x) = x^n - 2(-1)^n$ for all n. Indeed, we can show this using induction. At $n = 1$, this is clearly true. When $n \geq 1$, we can write

$$g_{n+1}(x) = xg_n(x) + (-1)^n(2x + 2) = x(x^n - 2(-1)^n) + (-1)^n(2x + 2) = x^{n+1} - 2(-1)^{n+1}.$$

Thus, this statement is true for all n. In particular, at $n = 2024$, we have that $g_{2024}(x) = x^{2024} - 2$ which has roots at $x = \pm \sqrt[2024]{2}$. This means that $f_{2024}(x)$ has roots at $x = -1 \pm \sqrt[2024]{2}$, so the answer is $-1 + 1 + 2 = \boxed{2}$.

Problem 2: A 11×11 square is divided into 121 unit squares, forming a 12×12 grid of vertices. How many ways can we color these vertices either red or blue such that each unit square has exactly 2 red vertices?

Solution. We claim that the answer is $2^{n+2} - 2$ for a $n \times n$ square. We say that a column has alternating colors if no two adjacent vertices in that column have the same color. We begin by showing that, if there exists a column that has alternating colors, then the rest of the columns must also have alternating colors.

Label the columns of vertices from 0 to n going from the left to the right, and suppose that column i has alternating colors. Consider the squares between columns i and $i + 1$. Since each of these squares already has exactly 1 red vertex, the squares dictate that every pair of adjacent vertices in column $i + 1$ must have differing colors. That is, column $i + 1$ also has alternating colors. We can continue this logic going left from column i and right from column $i + 1$ to see that every column has alternating colors as claimed.

Next, suppose that we fix the colors of the top row of vertices. We claim that, if there exists at least one pair of adjacent vertices in the top row that are identically colored, then the colors of the rest of the vertices are determined. If vertices in columns i and $i+1$ are identically colored, then it is easy to see that both columns i and $i+1$ have alternating colors. From the above result, we therefore know that every column has alternating colors. Since we know the color of a vertex in each column, we therefore know the colors of every vertex. This contributes $2^{n+1}-2$ colorings, since there are 2^{n+1} different ways to color the top row of vertices, and only 2 of those ways lacks a pair of identically colored vertices.

Next, note that we can apply the same logic on the first column of the grid instead of the first row of the grid to obtain another $2^{n+1}-2$ colorings when the first column does not have alternating colors. These colorings must be distinct from the previous colorings since we showed that every row has alternating colors, which precludes the cases from earlier. Lastly, the remaining case is when both the first row and first column have alternating colors, in which case the entire grid must be a checkerboard. This adds a final 2 colorings, producing $2^{n+2}-2=2^{13}-2=\boxed{8190}$ colorings in all.

Problem 3: Suppose you have a quadratic Q, defined by $y=\frac{x^2}{22}+22$. A circle C intersects Q at four points, the x-coordinates of three of these points being 7, -6, and -23. Find the coordinates of the last intersection point.

Solution. Let the circle C be defined by the equation $(x-x_0)^2+(y-y_0)^2=r^2$, and let $y=ax^2+c$ be any quadratic that intersects C at four points. The x-values of the intersection points are precisely the solutions to the equation

$$(x-x_0)^2+(ax^2+c-y_0)^2=r^2.$$

we can see that the LHS of this equation is a quartic polynomial, and that the coefficient of x^3 in this polynomial is 0. By Vieta's Formula, the sum of the solutions to the polynomial must be 0. Applying this logic to the quadratic Q yields that, if x' is the desired quantity, then $x'+7-6-23=0$, so $x=\boxed{22}$.