2025 DUKE MATH MEET GUTS ROUND SOLUTIONS

Set 1.

1. Consider the sequence $2, 0, 2, 5, 2, 0, 2, 5, 2, \dots,$ where the four numbers $2, 0, 2, 5$ repeat indefinitely. Let b_n be the sum of the first n terms of this sequence. Find the unique positive integer k such that $b_k = 2025$.

   Proposed by: Akshar Yeccherla

   Solution: Note that the sum of each block of four elements is $2 + 0 + 2 + 5 = 9$. To find when $b_k = 2025$, we find out how many complete blocks come before 2025. It turns out there are precisely $\frac{2025}{9} = 225$ blocks, at position $4 \cdot 225 = \boxed{900}$, we have that $b_{900} = 2025$.

2. Find the number of ordered triples of nonnegative integers (a, b, c) satisfying

$$a + \frac{b}{2} + \frac{c}{4} = 20.25$$

   Proposed by: Nikhil Pesaladinne

   Solution: Multiplying both sides by 4 gives the equivalent equation $4a + 2b + c = 81$. For fixed a and b, the value of c is forced to be $c = 81 - 4a - 2b$, so we need

$$4a + 2b \leq 81.$$

   Let a range from 0 to $\lfloor 81/4 \rfloor = 20$. For a given a, the allowable b satisfy

$$2b \leq 81 - 4a \implies b \leq \left\lfloor \frac{81 - 4a}{2} \right\rfloor = \lfloor 40.5 - 2a \rfloor = 40 - 2a,$$

   since $40.5 - 2a$ is always a half-integer. Thus, for each fixed a, there are

$$(40 - 2a) + 1 = 41 - 2a$$

   choices for b (from 0 up to $40 - 2a$), and c is then determined. Therefore, the total number of solutions is

$$\sum_{a=0}^{20} (41 - 2a) = 21 \cdot 41 - 2 \sum_{a=0}^{20} a = 861 - 2 \cdot \frac{20 \cdot 21}{2} = 861 - 420 = \boxed{441}.$$

3. The number 2025 has the special property that the fourth digit divides the second digit (5 divides 0) and the third digit divides the first digit (2 divides 2). How many four digit numbers greater than 1000 have this property? (**Note:** 0 does not divide 0)

   Proposed by: Akshar Yeccherla

   Solution: Let this four digit number be $\overline{abcd}$. Our answer is the product of the number of pairs (a, c) (such that c divides a, and $a \neq 0$) and (b, d) (such that d divides b).

   We can enumerate these pairs for (b, d). If $d = k$, then $b = 0, k, 2k, \dots$. Thus, for $d = k$, there are $1 + \left\lfloor \frac{9}{k} \right\rfloor$ possibilities. Summing over $d = 1, \dots, 9$ yields $10+5+4+3+2+2+2+2+2 = 32$ such pairs. To enumerate the pairs for (a, c), we can throw out all pairs with first element 0, which yields $32 - 9 = 23$ pairs. Multiplying yields $32 \cdot 23 = \boxed{736}$ such numbers.

Set 2.

1. The Duke Blue Devil's favorite numbers are of the form $\overline{DUKE}$, where D is non-zero and $\overline{DUKE}$ has the same number of factors as 2025. How many favorite numbers does he have?

   Proposed by: Nikhil Pesaladinne

   Solution: We need four-digit integers $N = \overline{DUKE}$ (i.e. $1000 \leq N \leq 9999$ with $D \neq 0$) that have the same number of positive divisors as 2025. Let $d(x)$ be the number of factors of x. Since
$$2025 = 81 \cdot 25 = 3^4 \cdot 5^2,$$
$$\implies d(2025) = (4+1)(2+1) = 5 \cdot 3 = 15.$$

   Hence we must count four-digit N with exactly 15 divisors. The divisor function $d(n)$ equals 15 precisely in the following prime–power shapes (because $15 = 15$ or $15 = 5 \cdot 3$):
$$N = p^{14} \quad \text{or} \quad N = p^4 q^2 \quad (p, q \text{ distinct primes}).$$

   There are no four-digit 14th powers, since $2^{14} = 16384 > 9999$, so only the p^4q^2 shape remains. List possible primes p with $p^4 \leq 9999$:
$$2^4 = 16, \quad 3^4 = 81, \quad 5^4 = 625, \quad 7^4 = 2401, \quad 11^4 = 14641 > 9999.$$

   For each $p \in \{2, 3, 5, 7\}$, choose a prime $q \neq p$ so that $N = p^4q^2$ is four-digit:

   - $p = 2$: $N = 16q^2$; need $1000 \leq 16q^2 \leq 9999 \iff 63 \leq q^2 \leq 624$, so $q \in \{11, 13, 17, 19, 23\}$ (5 values).
   - $p = 3$: $N = 81q^2$; need $1000 \leq 81q^2 \leq 9999 \iff 12.345 \leq q^2 \leq 123.43$, so $q \in \{5, 7, 11\}$ (3 values).
   - $p = 5$: $N = 625q^2$; need $1000 \leq 625q^2 \leq 9999 \iff 1.6 \leq q^2 \leq 15.998$, so $q \in \{2, 3\}$ (2 values).
   - $p = 7$: $N = 2401q^2$; need $1000 \leq 2401q^2 \leq 9999 \iff 0.416 \leq q^2 \leq 4.162$, so $q = 2$ (1 value).

   This in total gives us $\boxed{11}$ values.

2. In rectangle $DUKE$, diagonal $\overline{UE}$ is drawn along with altitudes $\overline{DX}$ of $\triangle EDU$ and $\overline{KY}$ of $\triangle UKE$. Given that $UK = 20$, and $UE = 25$, compute XY.

   Proposed by: Nikhil Pesaladinne

   Solution: We first note that $EK = DU = 16$ from Pythagorean Theorem. Since $\angle KUY = \angle KUE$ we get that triangles $\triangle KYU$ and $\triangle KEU$ are similar, along with triangles $\triangle DXE$ and $\triangle DUE$. From this, we get that $EX = UY = 9$, so $XY = 25 - 2 \cdot 9 = \boxed{7}$.

3. How many four digit numbers can be expressed as $\overline{DUKE}$, where D, U, K, E are not necessarily distinct integers and $0 < D, U, K, E < 10$ such that every permutation of its digits is divisible by 4 ($\overline{DUKE}$, $\overline{DEUK}$, etc).

   Proposed by: David Cong

Solution: For ease of notation, let the number be $\overline{abcd}$. For a number to be divisible by 4, the last two digits must form a two-digit number that is divisible by 4. It suffices to show that every two-digit combination of a, b, c, and d forms a two-digit number that is divisible by 4.

We can eliminate the possibility of odd integers for a,b,c,d. We are now left with 2,4,6,8. If an integer is repeated, we have $x + 10x \equiv 0 \pmod 4$ which is equivalent to $3x \equiv 0 \pmod 4$. This condition is only satisfied by 4 and 8.

For two distinct integers, they must satisfy both $10x + y \equiv 0 \pmod 4$ and $10y + x \equiv 0 \pmod 4$ or $2x + y \equiv 0 \pmod 4$ and $2y + x \equiv 0 \pmod 4$. This condition can not be satisfied by the selection of $x \equiv 2 \pmod 4$ and $y \equiv 2 \pmod 4$ or $x \equiv 2 \pmod 4$ and $y \equiv 0 \pmod 4$, therefore the only possible digits are 4 and 8. Each digit of $\overline{abcd}$ can either be a 4 or an 8, leading to $2^4 = \boxed{16}$ total possibilities.

Set 3.

1. Consider the tetrahedron $T = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \geq 0, x + y + z \leq 1\}$. How many points of the form $(\frac{a}{6}, \frac{b}{6}, \frac{c}{6})$, with a, b, and c integers, lie on the surface of the tetrahedron?

   Proposed by: Havish Shirumalla

   Solution: To count the number of such points in the entire tetrahedron, we are interested in integers $a, b, c \geq 0$ which satisfy $a + b + c \leq 6$. This is a sum of 7 different stars and bars, i.e. $\binom{2}{2} + \binom{3}{2} + \cdots + \binom{8}{2} = 84$.

   To find the grid points in the interior, we are interested in integers $a, b, c \geq 1$ which satisfy $a + b + c < 6$. Equivalently, setting $a' = a + 1, b' = b + 1, c' = c + 1$, we want nonnegative a', b', c' which satisfy $a' + b' + c' \leq 2$. Using stars and bars again, we have $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} = 10$. Thus, the final answer is the number of total points subtracted by the number of interior points, which is $84 - 10 = \boxed{74}$.

   Alternatively to compute these quantities, note that $a+b+c \leq 6$ is equivalent to $a+b+c+d = 6$, where d is a non-negative term that we can "throw away". Thus, by stars and bars, the total number of points is $\binom{9}{3} = 84$. Similarly, the number of interior points is $\binom{5}{3} = 10$.

2. In triangle ABC, we have that $AB = 10$, $AC = 9$, and $BC = 17$. Let point P lie on $\overline{AB}$ such that $AP = 4$ and point Q lie on $\overline{AC}$ such that $AQ = 3$. Let T be the intersection of $\overline{BQ}$ and $\overline{CP}$. Let D be the intersection of line AT and the line through B parallel to $\overline{AC}$. Find the area of quadrilateral $ABDC$.

   Proposed by: Erick Jiang

   Solution: From Heron's formula, we get the area of $\triangle ABC$ is 36. Also note that $\triangle ABC$ and $\triangle ADC$ share the same height and base, so their areas are equal. It remains to find the area of $\triangle ABD$. Let P be the intersection of $\overline{AD}$ and $\overline{BC}$. Note that $\triangle BDP$ is similar to $\triangle CAP$, which gives us that $\frac{[ABD]}{[ADC]} = \frac{BP}{PC}$.

   To find $\frac{BP}{PC}$, we assign a mass of 2 to point A, which gives us masses of $\frac{4}{3}$ and 1 for points B and C respectively. This gives us $\frac{BP}{PC} = \frac{3}{4} \implies \frac{[ABD]}{[ADC]} = \frac{3}{4} \implies [ABD] = 27$. To end, $[ABCD] = [ABD] + [ADC] = 27 + 36 = \boxed{63}$.

3. Alice and Bob are playing a game in which, on each turn, they must name a prime number. On the first turn, Alice names the number 2. On each turn, a player can add a positive integer to the current number to get another prime number, but they cannot add more than two times what is necessary to do so. In other words, if the current prime number is n, and the smallest prime number greater than n is p, the player can only add at most $2(p - n)$ to n to yield another prime number.

   Define game G_q, for positive prime q, such that the game ends when a player names q, and that player is declared the winner. Assuming optimal play from Alice and Bob, find the sum of all q such that Bob has a winning strategy in the game G_q.

   Proposed by: Leo Yang

   Solution: Alice will win if the winning number q is not 3. Notice that in any game G_q, Alice says 2, Bob says 3, and then Alice has a choice between 5 or 7. If 7 guarantees the winning strategy, Alice will say it, otherwise, Alice will say 5, which forces Bob to say 7, which guarantees the losing strategy to Bob, so Alice again has the winning strategy.

Unless the winning number is 3, Alice will always win, as Alice wins on 2, 5, 7, and any prime greater than 7. Thus, our answer is $\boxed{3}$.

Set 4.

1. You glue together equilateral, equal size triangles edge-to-edge into a hollow solid such that each edge is shared by exactly two triangles, and there are no holes/punctures. Suppose that around each vertex, either 5 triangles meet or 6 triangles meet, and there are no other possibilities. How many vertices have 5 triangles meeting around them?

   Proposed by: Havish Shirumalla

   Solution: Write T for the number of triangles, V for the number of vertices, V_6 for the number of 6-valent vertices, V_5 for the number of 5-valent vertices, and E for the number of edges. Note $V = V_5 + V_6$. (Here, a k-valent vertex is a vertex with k triangles meeting around it).

   Each triangle has 3 vertices, so $3T = 5V_5 + 6V_6$.

   Since each edge borders 2 triangles, $E = \frac{3T}{2}$. From the Euler characteristic, $V - E + T = 2$, we have that $V - \frac{3T}{2} + T = 2 \implies T = 2(V-2)$.

   Plugging into the first equation, $6(V-2) = 5V_5 + 6V_6 \implies 6V_5 + 6V_6 - 12 = 5V_5 + 6V_6 \implies V_5 = \boxed{12}$.

2. Square $ABCD$ with side length 4 has points E and F on sides $\overline{CD}$ and $\overline{AB}$, respectively, such that E is the midpoint of $\overline{CD}$ and $AF = 3$. Let G be a point on $\overline{EF}$ and H on $\overline{BC}$ such that $\triangle DEF \sim \triangle GHC$. If $\overline{EG}$ can be expressed as $\frac{a\sqrt{b}}{c}$, where b is not divisible by the square of any prime. Find $a+b+c$.

   Proposed by: Jaemin Kim

   Solution: First, note that quadrilateral $ECHG$ is a cyclic quadrilateral, as $\angle CEG + \angle CHG = (180° - \angle DEF) + \angle CHG = 180°$ by similarity.

   Importantly, we then know that $\angle GCH = \angle GEH = \angle EFD$. This implies that lines EH and DF are parallel.

   Drop a perpendicular from point E onto AB. Suppose the foot of this perpendicular is at point P on $\overline{AB}$, and the perpendicular intersects $\overline{DF}$ at K. Because $DF \parallel EH$ and $EP \parallel CB$, we have that $\overline{HB} = \overline{KP} = \frac{4}{3}$ from the simple triangle similarity $\triangle DEK \sim \triangle FPK$. Thus, $\overline{CH} = \frac{8}{3}$.

   From here, we compute for $\overline{GH}$ by using the given similarity $\triangle DEF \sim \triangle GHC$. $EF = \sqrt{17}$ from the pythagorean theorem on $\triangle EPF$, and this yields $\overline{GH} = \frac{16\sqrt{17}}{51}$. We are basically done. Since $\overline{EC} = 2$, we can calculate $\overline{EH} = \frac{10}{3}$. Then, by the Pythagorean theorem on $\triangle EGH$, $EG = \sqrt{(\frac{10}{3})^2 - (\frac{16\sqrt{17}}{51})^2} = \frac{38\sqrt{17}}{51}$, yielding a final answer of $\boxed{106}$.

3. You have a Christmas tree with 3 layers (top, middle, and bottom) and want to decorate it with 50 indistinguishable ornaments. How many ways can you distribute them across the three layers such that each layer has at least one ornament, and each layer has strictly more ornaments than the layer above it (i.e., the bottom layer has more ornaments than the middle layer, which has more ornaments than the top layer)?

   Proposed by: Jaemin Kim

Solution: Consider a similar problem, where the Christmas tree does not have restrictions on each layer having strictly more ornaments than the layer above it. The solution to this is simply stars and bars, $\binom{49}{2} = 1176$, where each layer has at least one ornament.

Now, we notice that this is equal to $6X + 3Y$, where X is the number of ways we can distribute the ornaments given the strictly decreasing condition and Y is the number of ways we can distribute the ornaments given that some two layers have the same number of ornaments. This idea uses the fact that a given arrangement of ornaments must either have a distinct number of ornaments in all layers, or have two layers with the same number of ornaments (all three layers cannot have the same number of ornaments, as $3 \nmid 50$. The coefficients in front of X and Y are due to the number of rearrangements possible across the layers.

To find Y, we can just count. We start with $\{2, 24, 24\}$ and end with $\{48, 1, 1\}$ for a total of 24. Solving for X in $1176 = 6X + 72$ yields $X = \boxed{184}$.

Set 5.

1. You have a ranked choice voting system with 17 candidates and k voters, where each voter has one ballot where they rank their 1st, 2nd, ... 16th, and 17th choices. Call a collection of ballots **super-non-condorcet** if there exists a permutation of all the candidates S1, S2... S17 such that S1 beats S2, S2 beats S3, ... S16 beats S17, S17 beats S1 (where A beats B if strictly more voters ranked A above B). Find the minimum value of k where there exists a **super-non-condorcet** collection of k ballots.

   Proposed by: Akshat Basannavar

   Solution: We claim the answer is $\boxed{3}$. Suppose k = 1. Trivially, the only ballot available determines the entire election. Since each individual ballot cannot have a cycle in preferences, there is not a 1-ballot super-non-condorcet collection. Now consider k = 2. Choose one ballot and consider the last-ranked candidate. For the collection of ballots to be super-non-condorcet, this candidate must beat at least one other candidate. But this is impossible, since the candidate is ranked below everyone else in one ballot, so an additional ballot would, at best, tie this candidate with everyone else. It suffices to prove there exists a super-non-condorcet collection of ballots for k = 3. Consider the following:

   Ballot 1: $\{1, 2, ..., 17\}$

   Ballot 2: $\{2, 3, ..., 17, 1\}$

   Ballot 3: $\{3, 4, ..., 17, 1, 2\}$

   Consider every pair of candidates $(i, i+1)$ where number 18 refers to candidate 1. Clearly, candidate $i$is ranked higher than candidate $i+1$ in at least two of the three ballots, so we have a cycle where 1 beats 2, 2 beats 3, and so on until 17 beats 1. Hence, we are done.

2. Let x_n, y_n, for $n \geq 0$, be infinite sequences of positive numbers such that $x_n = c \cdot y_n$ for some constant c, and $x_n^2 + (y_0 + y_1 + ... + y_n)^2 = 2500$ for all n. If $x_0 = 50 - 2^{-50}$, find

   $$\left\lfloor \sum_{i=0}^{\infty} x_i \cdot y_i \right\rfloor$$

   Proposed by: Akshat Basannavar

   Solution: Consider a circle of radius 50, and placing a rectangle that touches the interior of the circle with length x_0 and height y_0, one of whose vertices is the center of the circle. The equation $x_n = c \cdot y_n$ should imply all future rectangles with dimensions (x_i, y_i) are all similar to each other. Furthermore, by "stacking" these rectangles such that one of their sides all lie on the same line passing through the center of the circle, we can achieve exactly the equation

   $$x_n^2 + (y_0 + y_1 + ... + y_n)^2 - 2500$$

   for all n.

   Now that we have geometrically interpreted this problem, notice that $x_0 = 50 - 2^{-50}$ essentially means we are tiling a quarter of the circle with negligibly thin rectangles. The desired sum calls for the floor of the sum of the areas of all rectangles. Since the circle has radius 50, we simply find the area of the quarter-circle, which is $\frac{50^2\pi}{4} \approx 625(3.1415) = 1963.49$, so our answer is $\boxed{1963}$.

3. A permutation p of length n is an arrangement of the integers $1, 2, 3, \ldots, n$. For example, $(3, 1, 2)$ and $(1, 3, 2)$ are permutations of length 3. If the ith number in the permutation is i, then we call i a fixed point. For example, in $(1, 3, 2)$, 1 is the only fixed point. How many permutations $(p_1, p_2, \ldots, p_{12})$ of length 12 are there with three fixed points, such that $p_1 = 3, p_2 = 6, p_3 = 4$?

   Proposed by: Akshar Yeccherla

   Solution: One can do this with careful casework, but a cleaner solution is to use the cycle decomposition. Currently, we have a cycle $(1, 3, 4, \ldots)(2, 6, \ldots)$. None of these five points can be fixed, so we have to choose them from the remaining 6 in $\binom{7}{3} = 35$ ways. Afterwards, we have four numbers left.

   We have 4 numbers left to arrange. They can either be assigned in the existing cycles or their own, but cannot have fixed points (these are called derangements, and are easy to compute for small numbers using Principle of Inclusion-Exclusion). We select some to be in their own derangement, and some to be in the cycles, and case on this instead

   (number of ways to pick numbers in derangement) $*$ (number of derangements) * (number of ways to permute remaining numbers) $*$ (number of ways to assign to cycles) $*$ (number of ways to close cycles)

   Zero in derangement: $\binom{4}{0} \cdot 1 \cdot 4! \cdot 5 \cdot 2 = 240$

   Two in derangement: $\binom{4}{2} \cdot 1 \cdot 2! \cdot 3 \cdot 2 = 72$

   Three in derangement: $\binom{4}{3} \cdot 2 \cdot 1! \cdot 2 \cdot 2 = 32$

   Four in derangement: $\binom{4}{4} \cdot 9 \cdot 0! \cdot 1 \cdot 2 = 18.$

   Thus, the answer is $35 \cdot (240 + 72 + 32 + 18) = \boxed{12670}$.

Set 6.

1. Akshar wants to build Pokémon gyms at the 3 northernmost state capitals in the continental US. What is the area in mi^2 of the triangle formed by these three Pokémon gyms?

   Proposed by: Nikhil Pesaladinne

   Solution: $\boxed{12122}$

2. Jaemin has just discovered that Spiritombs are real! Given that every Homo sapiens in history became a Spiritomb when they passed away and that Spiritombs are uniformly distributed on the surface of the Earth, how many Spiritombs live in Jaemin's lawn (area: 1 mi^2)?

   Proposed by: Nikhil Pesaladinne

   Solution: $\boxed{548}$

3. Nikhil recently found out that Pikachu releases the same electrical output as 100 adult electric eels. How many Pikachus would you need to power the average U.S. home for 24 hours?

   Proposed by: Nikhil Pesaladinne

   Solution: $\boxed{2000}$