2025 DUKE MATH MEET INDIVIDUAL ROUND SOLUTIONS

1. Thomas wants to go Go-Karting, but forgot the name of the Go-Kart place. He knows that it consists of two letters of the 26-letter English alphabet (any letter from A-Z). He also knows that the second letter does not appear earlier in the alphabet than the first letter. For example, DU, and MU are possible names, but NC is not. How many possible names are there for the Go-Kart place?

   Proposed by: Akshar Yeccherla

   Solution: We can casework on the first letter. If it is A, then the second letter has 26 possibilities, if it is B, then the second letter has 25 possibilities, and so on. Generally, for the ith letter, there are $27 - i$ possibilities. Thus, our answer is

$$1 + 2 + 3 + \cdots + 25 + 26 = \frac{26 \cdot 27}{2} = \boxed{351}$$

2. How many integers from 1 to 2025, inclusive, can be expressed in the form $\lfloor 2x \rfloor + \lfloor 0x \rfloor + \lfloor 2x \rfloor + \lfloor 5x \rfloor$ for some positive real number x?

Proposed by: David Cong

Solution: Let us simplify the original expression to $2\lfloor 2x \rfloor + \lfloor 5x \rfloor$. Let $x = n + m$ where n is an integer and $0 \le m < 1$. The original expression can be rewritten as $9n + 2\lfloor 2m \rfloor + \lfloor 5m \rfloor$. The floor function can be broken into 6 subcases:

Case 1: $0 \le m < 0.2$: $9n + 0 + 0 = 9n$

Case 2: $0.2 \le m < 0.4$: $9n + 0 + 1 = 9n + 1$

Case 3: $0.4 \le m < 0.5$: $9n + 0 + 1 = 9n + 2$

Case 4: $0.5 \le m < 0.6$: $9n + 2 + 2 = 9n + 4$

Case 5: $0.6 \le m < 0.8$: $9n + 2 + 3 = 9n + 5$

Case 6: $0.8 \le m < 1$: $9n + 2 + 4 = 9n + 6$

In each block of 9 consecutive integers, 6 can be expressed. In the interval 1 to 2025 there are $2025/9 = 225$ blocks of 9 integers. Therefore, $6 \times 225 = \boxed{1350}$ integers in the interval can be expressed by the original equation.

3. Consider triangle ABC. Let $v(A), v(B), v(C)$ be (possibly negative) real number values assigned to the vertices of A, B, C. The value of an edge is the sum of the values of its endpoints, and the capacity of an edge AB, denoted $c(AB)$, is the maximum allowed value of edge AB. If $c(AB) = 11, c(BC) = 8, c(AC) = 25$, what is the largest possible value of $v(A) + v(B) + v(C)$?

   Proposed by: Akshar Yeccherla

   Solution: The three inequalities we get are

$$v(A) + v(B) \leq c(AB)$$

$$v(B) + v(C) \leq c(BC)$$

$$v(A) + v(C) \leq c(AC)$$

   Adding and dividing by two yields

$$v(A) + v(B) + v(C) \leq \frac{c(AB) + c(BC) + c(AC)}{2}$$

   for which the maximum is always achievable as we have a system of three variables and three equations, specifically $(A, B, C) = (14, -3, 11)$. Thus, our answer is $\boxed{22}$.

4. Fix set $S = \{-3, -2, -1, 0, 1, 2, 3\}$. Let $P(x, y)$ be a real polynomial in two variables with both the degree of x and the degree of y equal to 6. Assume that $P(a, b) = 0$ for all pairs (a, b) such that $a \in S$ and $b \in S$, except possibly $(0, 0)$, and suppose the coefficient of x^6y^6 is 1. Compute $P(0, 0)$.

   Proposed by: Havish Shirumalla

   Solution: For each $a \neq 0$, the polynomial $P(a, y)$ has 7 zeroes and has degree 6, so it must be the zero polynomial. Therefore, the polynomial $U(x) = (x+3)(x+2)(x+1)(x-1)(x-2)(x-3)$ divides $P(x, y)$.

   Similarly, the polynomial $V(y) = (y + 3)(y + 2)(y + 1)(y - 1)(y - 2)(y - 3)$ must divide $P(x, y)$. Since the degrees of U and V are 6, we must have $P(x, y) = cU(x)V(y)$. Note that c must be 1 since we are given that the coefficient of x^6y^6 is 1. Thus, $P(0, 0) = U(0)V(0) = (3 \cdot 2 \cdot 1 \cdot (-1) \cdot (-2) \cdot (-3))^2 = (-36)^2 = \boxed{1296}$.

5. Let $A = (0, 0)$ and $B = (20, 0)$. Define ω to be the circle consisting of all points X satisfying

$$\frac{XA}{XB} = \frac{3}{2}.$$

Let γ be the circle with diameter $\overline{AB}$. Circles ω and γ intersect at distinct points M and N. The length MN can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Nikhil Pesaladinne

Solution: Let $A = (0, 0)$ and $B = (20, 0)$. The circle γ with diameter $\overline{AB}$ is

$$\gamma : \ (x - 10)^2 + y^2 = 10^2.$$

It is clear the diameter of γ lies on the x-axis. We can also check that the points $(12, 0)$ and $(60, 0)$ also lie on ω via confirming the ratio relations:

$$\frac{XA}{XB} = \frac{12}{8} = \frac{3}{2}, \qquad \frac{XA}{XB} = \frac{60}{40} = \frac{3}{2}.$$

Hence $(12, 0)$ and $(60, 0)$ lie on ω, so ω is the circle with these points as endpoints of a diameter. Therefore ω has center

$$W = \left(\frac{12 + 60}{2}, 0\right) = (36, 0)$$

and radius $r = \dfrac{60 - 12}{2} = 24$, i.e.

$$\omega : \ (x - 36)^2 + y^2 = 24^2.$$

Let M and N be the intersections of γ and ω. Because both centers $G = (10, 0)$ and $W = (36, 0)$ lie on the x-axis, the common chord MN is perpendicular to the x-axis; hence it is a vertical line $x =$ constant. Subtracting the two circle equations eliminates y and gives that x–coordinate:

$$(x - 10)^2 + y^2 = 100, \qquad (x - 36)^2 + y^2 = 576.$$

$$(x - 10)^2 - (x - 36)^2 = 100 - 576 = -476.$$

$$\big((x - 10) - (x - 36)\big)\big((x - 10) + (x - 36)\big) = 26(2x - 46) = 52x - 1196 = -476,$$

so

$$52x = 720 \quad \Longrightarrow \quad x = \frac{720}{52} = \frac{180}{13}.$$

Thus the common chord is the vertical line $x = \dfrac{180}{13}$. Dropping an altitude from M (or N) to the x-axis, its y-coordinate satisfies the γ equation:

$$\left(\frac{180}{13} - 10\right)^2 + y^2 = 100 \Longrightarrow \left(\frac{50}{13}\right)^2 + y^2 = 100 \Longrightarrow y^2 = 100 - \frac{2500}{169} = \frac{14400}{169}.$$

Hence $|y| = \dfrac{120}{13}$, so the chord length is

$$MN = 2|y| = \frac{240}{13}.$$

Therefore $MN = \dfrac{m}{n} = \dfrac{240}{13} \implies \boxed{253}$

6. Let p, q be primes and r a positive integer such that

$$1 + p = q^r$$

For a fixed value p, we can express $p - 1$ as the product of primes as follows

$$p - 1 = 2 \times 3 \times 107 \times 6361 \times 69431 \times 20394401 \times 28059810762433$$

for which there exists a unique solution (q, r). Find $q + r$.

Proposed by: Akshar Yeccherla

Solution: Since $p - 1$ is even, then $1 + p$ is even. As q is prime, the only possibility for it is 2. Thus, it suffices to find r such that $1 + p = 2^r$ for the given value of $p - 1$.

Taking both sides of the equation modulo p, we have that

$$2^r \equiv 1 \pmod{p}$$

Then, let $\text{ord}_p(2)$ be the smallest positive integer such that k such that $2^k \equiv 1 \pmod{p}$. We must have that $\text{ord}_p(2) \mid r$. Furthermore, since $1 + p = 2^r$, then r must precisely be $\text{ord}_p(2)$

By Fermat's Little Theorem, we also have that $2^{p-1} \equiv 1 \pmod{p}$, thus $\text{ord}_p(2) \mid p - 1 \implies r \mid p - 1$. Note that the first several factors of $p - 1$ are $1, 2, 3, 6, 107, 214 \ldots$. The first four of these numbers are too small to be $\text{ord}_p(2)$, as p is a large number, thus $\text{ord}_p(2) \geq 107$.

Since we know that $1 + p = 2^r$, we can try to bound r by looking at $\log_2(p + 1) \approx \log_2(p - 1)$. Using $2^{10} = 1024 \approx 1000$ as a heuristic, we can obtain a loose bound $90 < r < 120$. Thus, $\text{ord}_p(2) = 214$ is too big, and $\text{ord}_p(2)$ must be 107. Thus, $r = 107$, and $q + r = 2 + 7 = \boxed{109}$.

7. Call a positive integer *sendy* if it consists of only the digits 6 and 7. Leo plays the following game with this number. He can choose to either delete the first digit (given it has at least one digit) and score 1 point, or delete the first two digits if they are exactly 67 (given it has at least two digits), and score 3 points. The game ends when the number has no more digits left. The **score** of a *sendy* number is the maximum number of points Leo can achieve while playing this game.

   For example, the **score** of 6767 is 6 points because Leo can delete 67 twice, for 3 points each. What's the largest possible **score** for a *sendy* number that has 500 digits and is divisible by 11?

   Proposed by: Leo Xu

   Solution: Consider the divisibility rule for 11 where if we index the first digit at 0, then sum of the digits at the even indices minus the sum of the digits at the odd indices must be divisible by 11.

   Then, a 67 with the 6 in an even index slot is $-1 \pmod{11}$ and a 67 with the 6 in an odd index slot is $+1 \pmod{11}$. First we test if 250 67's in a row works, but -250 mod 11 is not 0. So then if there's 249 67's and two other numbers we can use 124 67's in a row followed by a 6, followed by 125 67's in a row, then followed by a 7.

   Intuitively we can get this by saying that the first 124 67's is $-124 \pmod{11}$, then the 125 67's are $+125 \pmod{11}$. Then the 6 is on an even index and 7 on an odd one, so this nets out to $-1 \pmod{11}$. Then $-124 + 125 - 1 = 0 \pmod{11}$. The **score** of this number is $249 \cdot 3 + 2 = \boxed{749}$. To show maximality, note that the largest possible score is 750 (with 25 67's in a row, which we have shown is not possible as it is not divisible by 11).

8. You have 8 indistinguishable black chocolate wafer discs and 4 indistinguishable white cream filling discs. When you arrange these discs in a random order (with each possible arrangement equally likely), you can sometimes form a "complete oreo" - defined as a non-overlapping pattern "BWB" when reading from left to right.

   For example:

   - The pattern "BWB" counts as one complete oreo.
   - BWBWB counts as one complete oreo (not two, as the patterns overlap), and so does BWBBWW.
   - We want to count the maximum number of oreos: BWBWBWB counts as two complete oreos and not one.

   If all 12 pieces are arranged in a random order, then the expected number of complete oreos that will appear can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   Proposed by: Jaemin Kim

   Solution: We use the Principle of Inclusion-Exclusion. First, we count the number of BWB sequences. If we index the positions of the discs as 1 to 12, then the BWB sequence can start at the indices 1 to 10. Without loss of generality, let us assume it starts at index 1. Then, indices 1 to 3 are fixed. There are then $\binom{9}{3}$ ways to order the remaining 6 black discs and 3 white discs. By symmetry, this is the same for all 10 starting locations, thus there are $10\binom{9}{3}$ BWB sequences across all orders.

   However, we count the sequence $BWBWB$ as two oreos, when it should only be counted as 1, thus we overcount these type of sequences by exactly one and must subtract them. By similar logic to above, there are 8 starting locations and $\binom{7}{2}$ ways to order the remaining discs, for a total of $8\binom{7}{2}$ $BWBWB$ sequences.

   Finally, we count the sequence $BWBWBWB$ as one oreo, as we add three BWB sequences and subtract two $BWBWB$ sequences. This should yield two oreos, so we must add the count the $BWBWBWB$ sequences. by similar logic, there are $6\binom{5}{1}$ sequences.

   Finally, we count the sequence $BWBWBWBWB$ as three oreos, as we add four BWB sequences, subtract three $BWBWB$ sequences, and add two $BWBWBWB$ sequences. However, this should count as two oreos, so we overcount these by one oreo and must subtract these sequences from the count. There are $4\binom{3}{0}$ such sequences.

As there are $\binom{12}{4}$ total ways to order the discs, the expected number of oreos is

$$\frac{10\binom{9}{3} - 8\binom{7}{2} + 6\binom{5}{1} - 4\binom{3}{0}}{\binom{12}{4}} = \frac{698}{495}$$

Thus, our answer is $698 + 495 = \boxed{1193}$.

Note that this inclusion-exclusion logic is generalizable (and can thus be used to solve the problem for more discs), we wrote out all cases for the sake of clarity.

9. The cubic polynomial $x^3 - 20x^2 + bx - 125 = 0$ has three real roots that are consecutive terms in a geometric sequence. Find b.

   Proposed by: Jaemin Kim

   Solution: Suppose the three real roots of the polynomial can be written as a, ar, and ar^2. Then, by Vieta's, we have that $a+ar+ar^2 = 20$, $b = a \cdot ar + ar \cdot ar^2 + a \cdot ar^2 = ar(a+ar+ar^2)$, and $a \cdot ar \cdot ar^2 = a^3r^3 = 125$.

   Since the roots are real, we know that $ar = (125)^{\frac{1}{3}} = 5$. So $b = ar(a+ar+ar^2) = 5 \cdot 20 = \boxed{100}$.

10. In trapezoid $ABCD$, $\angle A = \angle D = 90°$. Let M and N be the midpoints of diagonals $\overline{AC}$ and $\overline{BD}$, respectively. Let Q and R be the other intersections of $\overline{BC}$ and the circles that go through points A, B, N and C, D, M respectively. Denote P as the midpoint of $\overline{QR}$, L as the midpoint of $\overline{BC}$, and T as the foot of the altitude from B to $\overline{CD}$. Let K be the midpoint of $\overline{MN}$. If $KL = 4$ and $PL = 2$, $\frac{TC}{BC}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by: Nikhil Pesaladinne

Solution: Without loss of generality, assume $AB < CD$. Note that N is the midpoint of BD, so $AN = NB = ND$ because $\angle A = 90°$. Analogously, $MA = MD = MC$.

By power of a point, $LB \cdot LQ = LN^2$ and $LC \cdot LR = LM^2$. Note L lies on the perpendicular bisector of AD, so $BL = CL$. Hence

$$2|LK| \cdot |MN| = |(LM + LN)| \cdot |(LM - LN)| = |LM^2 - LN^2| = LC|LR - LQ|$$

Since $PL = 2$, L is the midpoint of BC and P is the midpoint of QR, it follows that $|LR - LQ| = 4$. Hence $8|MN| = 2|BC|$. Note that $MN = \frac{|AB-CD|}{2} = \frac{|CT|}{2}$. Thus, $\frac{TC}{BC} = \frac{1}{2} \implies \boxed{3}$.