2025 DUKE MATH MEET RELAY ROUND SOLUTIONS

1 Relay Round - Set 1

1. Barbie the Bee is collecting honey from 100 distinct flowers labeled from 1 to 100. She visits a given flower if and only if all the digits of the flower are distinct and the flower is not even-numbered. How many flowers does she visit?

   Proposed by: Jaemin Kim

   Solution: We consider the labels from 1 to 100. The flowers labeled 1–9 all have distinct digits. Barbie only visits odd-numbered flowers, so among these she visits $1, 3, 5, 7, 9$ for a total of 5 flowers. Let the tens digit be a and the units digit be b. We must have

   $$a \in \{1, 2, \ldots, 9\}, \quad b \in \{1, 3, 5, 7, 9\}, \quad a \neq b$$

   and the number must be odd, so b is one of 5 odd digits. For each choice of b, there are 9 possible tens digits from 1 to 9, except we must exclude $a = b$, leaving 8 choices. Thus the number of such two-digit flowers is $5 \cdot 8 = 40$. The label 100 is even and also has repeated digits, so it is not visited. Therefore the total number of flowers Barbie visits is $5 + 40 = \boxed{45}$.

2. Let $T =$ TNYWR. All bees in a certain colony have body patterns made up of n stripes, where n is the remainder when T is divided by 10. Each stripe is black or yellow, and every bee's stripe pattern is not equal to its reverse. What's the maximum number of unique stripe patterns in such a bee colony?

   (e.g. (BYBY) is a good stripe pattern as (YBYB) is its reverse, but (YBBY) is not, as (YBBY) is its reverse)

   Proposed by: Jaemin Kim

   Solution: From the first problem, we have $T = 45$, so the remainder when T is divided by 10 is

   $$n \equiv T \pmod{10} \implies n = 5.$$

   Thus each stripe pattern is a string of length 5 using the letters B (black) and Y (yellow). There are $2^5 = 32$ total possible stripe patterns of length 5. A pattern is forbidden if it is equal to its reverse, i.e., it is a palindrome. To count palindromes of length 5, label the positions $1, 2, 3, 4, 5$. For a palindrome we must have

   $$\text{position } 1 = \text{position } 5, \quad \text{position } 2 = \text{position } 4,$$

   and position 3 can be anything. Thus:

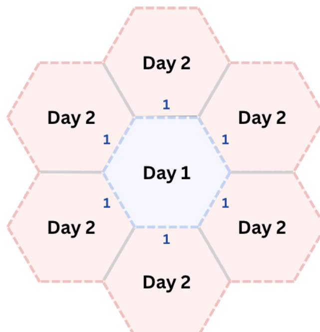
   $$\text{choices: } 2 \text{ for position } 1,\ 2 \text{ for position } 2,\ 2 \text{ for position } 3,$$

   giving $2^3 = 8$ palindromic patterns. Therefore the number of allowed stripe patterns (those not equal to their reverse) is

   $$32 - 8 = \boxed{24}.$$

3. Let $T =$ TNYWR. Barbie the Bee is building a honeycomb structure. On day 1, she builds a hexagonal honeycomb with side length 1. On each subsequent day, she covers the entire perimeter of the existing honeycomb with hexagons of side length 1. The resulting structure after day 2 is shown below. What's the perimeter of Barbie's honeycomb after T days (the perimeter is just the length of the exterior of the entire honeycomb)?

   (i.e. after Day 1, the honeycomb will have a perimeter of 6 while after Day 2, the honeycomb will have a perimeter of 18)

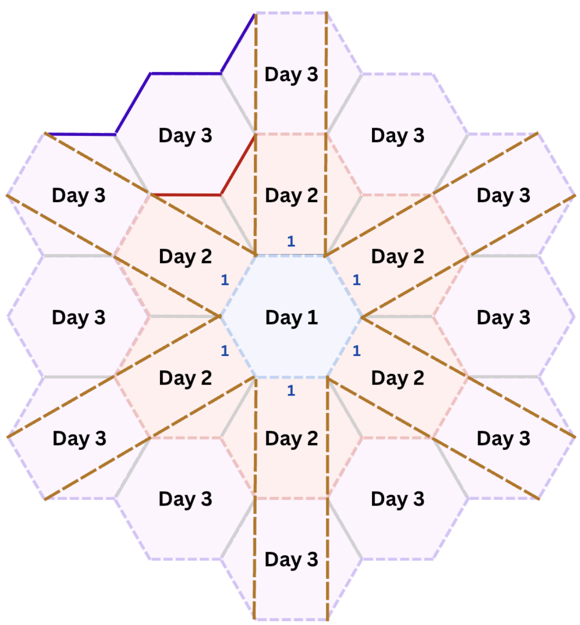


Barbie's honeycomb after 2 days

Proposed by: Jaemin Kim

Solution: We have $T = 24$.

First, extend the honeycomb for one more day in order to notice a pattern.



Barbie's honeycomb after 3 days

Then, as shown in the figure, draw 12 dashed (yellow) lines from the Day 1 middle hexagon out towards the perimeter. We notice that these lines partition the perimeter of any honeycomb structure after Day 1 into 12 parts. There are 6 parts of unit length, and 6 other zig-zagy parts. It is easy to notice and prove that the length of the zigzagy parts increases by 2 units each day.

Thus, the perimeter after T days is $6 \cdot 1 + 6 \cdot 2 \cdot (T - 1) = 12T - 6$. For $T = 24$ this yields $\boxed{282}$.

2 Relay Round - Set 2

1. In triangle DEF, there is a point U on $\overline{DF}$ and a point K on $\overline{EF}$ such that K is the midpoint of $\overline{EF}$. If the area of quadrilateral $DUKE$ is 12 and the area of triangle KUF is 2, the ratio $\frac{DF}{DU}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

   Proposed by: Jaemin Kim

   Solution: Draw $\overline{UD}$. Since K is the midpoint of $\overline{EF}$, the segment $\overline{UK}$ is a median in $\triangle EUF$, so it bisects its area:

$$[\triangle UKE] = [\triangle UKF] = 2 \quad \Longrightarrow \quad [\triangle EUF] = 2 + 2 = 4.$$

   We now have two triangles with the same height, $\triangle DUE$ and $\triangle UEF$, with areas 10 and 4 respectively. Since they have the same height, the ratio of their areas must be the ratio of their bases, which gives us that $\frac{DU}{UF} = \frac{5}{2} \implies \frac{DF}{DU} = \frac{7}{5} \implies \boxed{12}$.

2. Let T = TNYWR, and $p = \frac{T-2}{T+2}$. Duke is in a close basketball game with their rival UNC. The current score is 67-66 in Duke's favor, and there are 3 minutes left in the game. After each minute, Duke outscores UNC by 1 point with probability p and UNC outscores Duke by 1 point with probability $1-p$. Given that each scoring update is independent of all previous scoring updates, the probability that Duke is winning when the game is over can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $1000m + n$.

   Proposed by: Nikhil Pesaladinne

   Solution: With $T = 12$, we have

$$p = \frac{T-2}{T+2} = \frac{12-2}{12+2} = \frac{10}{14} = \frac{5}{7}, \qquad q = 1 - p = \frac{2}{7}.$$

   Let each of the remaining 3 minutes be a step of $+1$ (Duke outscores UNC) with probability p or -1 with probability q. Duke's current lead is $+1$. After 3 steps, the final lead is $1 + S_3$, where $S_3 \in \{-3, -1, 1, 3\}$. Duke is winning at the end iff $1 + S_3 > 0$, i.e. $S_3 \in \{1, 3\}$, which means Duke wins at least 2 of the 3 minutes. Thus

$$\mathbb{P}(\text{Duke wins}) = \binom{3}{2} p^2 q + p^3 = 3\left(\frac{5}{7}\right)^2 \left(\frac{2}{7}\right) + \left(\frac{5}{7}\right)^3 = \frac{150}{343} + \frac{125}{343} = \frac{275}{343}.$$

$$\Longrightarrow \ 1000m + n = 1000 \cdot 275 + 343 = \boxed{275343}.$$

3. Let T = TNYWR. Let K be the units digit of T. The Duke Blue Devil mascot loves to stack donuts on the three prongs of his trident. If he receives K distinct donuts, how many ways can he arrange them on his trident? Note that the order in which he places the donuts on each prong matters.

   Proposed by: Nikhil Pesaladinne

   Solution: Given $T = 275343$, the units digit is $K = 3$. We need the number of ways to place 3 distinct donuts onto 3 prongs where order on each prong matters. This is equivalent to splitting a permutation of the 3 donuts into three (possibly empty) ordered lists. Take any permutation of the 3 donuts (3! ways), then insert two identical dividers to split the line into three consecutive blocks (one per prong). Allowing empty blocks, the number of divider placements is

$$\binom{K+2}{2} = \binom{3+2}{2} = \binom{5}{2} = 10.$$

   Hence the total number of arrangements is

$$3!\binom{5}{2} = 6 \cdot 10 = \boxed{60}.$$