2025 DUKE MATH MEET TEAM ROUND

1. Pauline is traveling to the center of the earth in a ship. The path consists of 2400 miles of rock and 1600 miles of magma, and her ship travels at a constant positive **integer** speed in miles per hour on rock, and a different constant positive **integer** speed in miles per hour on magma. Given that her ship spent h hours traversing rock, and h hours traversing magma, for positive integer h, what is the largest possible amount of time her trip took in total, in hours?

2. Right triangle ABC has right angle at B. Three circles are centered at A, B, and C, initially with radius 0. At time $t_A = 6$ seconds, circle A's radius starts increasing at 1 unit per second. At time $t_B = 10$ seconds, circle B's radius starts increasing at 1 unit per second. At some unknown time t_C seconds, circle C's radius starts increasing at 1 unit per second. Given that AB is 12 units long, and each pair of the three circles intersect at the exact same time, find the length BC in units.

3. A robot has 5 rules under which you are allowed to talk to it:

   (a) You have an interesting question for it.

   (b) It has an interesting question for you.

   (c) You are talking about DMM problems.

   (d) You may ask to add a rule.

   (e) You may ask it about Putnam questions

   Each time you ask to add a rule, a new rule gets added. Every time you and the robot talk, the robot writes down a letter based on the type of conversation. Each day for three days you and the robot talk once. We'll call the potential added rules (f) and (g). As a result, the robot has written down a three-letter string. How many different possibilities are there for this string?

   For example, if we talked about an interesting question you have, and then about adding a new rule, and then about the new rule, the string *adf* would be written down.

4. Let rectangle $ABCD$ have side lengths $AB = 8$ and $BC = 12$. Point P lies outside rectangle $ABCD$ such that segment $\overline{PA}$ bisects $\overline{BC}$ at M and $PM = 6$. If the value of $PB * PC$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

5. The number 2025 has the interesting property that it is divisible by its sum of digits. This can be generalized by considering the sum of digits in different bases (assuming digits can be greater than 9). As an example, the sum of digits for 2025 in base 100 is

$$20 + 25 = 45$$

   because $2025 = 20 \cdot 100 + 25$. Find the sum of all positive integers n such that n is divisible by its sum of digits in every base $b \geq 2$.

6. Let w be a complex number such that $a = 26$ is the smallest positive integer such that $w^a = 1$. Find the number of 26-tuples of 0s and 1s, $(s_1, s_2, ..., s_{26})$, such that $\sum_{n=1}^{26} s_n w^n = 0$.

7. Consider the ordered set of numbers $\{1, 2, 3, ..., n\}$. Let s_i, where $1 \le i \le n-1$ be the operation at position i which swaps the number at position i with the number at position $i+1$. Any sequence of s_i's is called a word. For example, let $n = 5$. The word (s_2) turns $\{1, 2, 3, 4, 5\}$ into $\{1, 3, 2, 4, 5\}$, and (s_3, s_4, s_3) turns $\{1, 2, 3, 4, 5\}$ into $\{1, 2, 5, 4, 3\}$. The *length* of a permutation is the number of operations in the shortest possible word which results in that permutation. Let $n = 100$. Over all permutations, what is the maximum *length* of a permutation?

8. Consider polynomial $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 11$, with roots r_1, r_2, r_3, r_4. For each permutation σ of the set $\{1, 2, 3, 4\}$, define

$$U(\sigma) = r_{\sigma(1)} r_{\sigma(2)} + r_{\sigma(3)} r_{\sigma(4)}.$$

If a permutation σ is chosen uniformly at random, the expected value of $U(\sigma)^2$ can be expressed as a common fraction $\frac{m}{n}$ in lowest terms. Find $m + n$.

9. Let X be a set of 12 distinct real numbers. The set X_c is obtained by adding a real number $c \ge 0$ to each element in X (formally, $X_c = \{x + c : x \in X\}$).

For each $c \ge 0$ such that X and X_c have no elements in common (formally, $X \cap X_c = \emptyset$), define $Y = X \cup X_c$. Sort Y in ascending order. Define binary string S_c such that for $1 \le k \le 24$, the kth digit of S_c is 1 if the kth element of Y is in X_c, and 0 otherwise. Let S_X be the set of all such strings S_c.

Over all possibilities of X, what is the maximum number of distinct strings in S_X?

10. Let I be the intersection of the angle bisectors of triangle ABC and let ray $\overline{AI}$ meet the circle defined by the points A, B, C at D. Denote the feet of the perpendiculars from I to lines $\overline{BD}$ and $\overline{CD}$ by E and F, respectively. If $IE = 2$, $IF = 3$, and $AD = 10$, the greatest possible length of EF^2 can be expressed as $a + b\sqrt{c}$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.