2025 DUKE MATH MEET TEAM ROUND SOLUTIONS

1. Pauline is traveling to the center of the earth in a ship. The path consists of 2400 miles of rock and 1600 miles of magma, and her ship travels at a constant positive **integer** speed in miles per hour on rock, and a different constant positive **integer** speed in miles per hour on magma. Given that her ship spent h hours traversing rock, and h hours traversing magma, for positive integer h, what is the largest possible amount of time her trip took in total, in hours?

   Proposed by: Akshar Yeccherla

   Solution: We use the formula $d = rt$ (distance = rate × time). Let r_{rock} be Pauline's speed on rock, and r_{magma} be Pauline's speed on magma. We can set up the equations

$$2400 = r_{\text{rock}}h \implies h = \frac{2400}{r_{\text{rock}}}$$

$$1600 = r_{\text{magma}}h \implies h = \frac{1600}{r_{\text{magma}}}$$

   We have that $r_{\text{rock}}, r_{\text{magma}}, h$ are integers. Then, the largest value we can set for h is the greatest common divisor of 2400 and 1600, which is $h = 800$ hours. This is achieved by $r_{\text{rock}} = 3$ miles per hour and $r_{\text{magma}} = 2$ miles per hour. The total time taken on the trip is $2h$, thus, our answer is $2h = \boxed{1600}$ hours.

2. Right triangle ABC has right angle at B. Three circles are centered at A, B, and C, initially with radius 0. At time $t_A = 6$ seconds, circle A's radius starts increasing at 1 unit per second. At time $t_B = 10$ seconds, circle B's radius starts increasing at 1 unit per second. At some unknown time t_C seconds, circle C's radius starts increasing at 1 unit per second. Given that AB is 12 units long, and each pair of the three circles intersect at the exact same time, find the length BC in units.

   Proposed by: Akshar Yeccherla

   Solution: We know that $AB = 12$ units. At time $t = 10$ seconds, A has expanded to radius 4, while B has radius 0. At this point, both A and B are expanding, thus the remaining distance of 8 units closes by 2 units per second. Thus, at time $t = 14$ seconds, both circles intersect (A has radius 8, B has radius 4).

   Thus, we need C to intersect A and intersect B at time $t = 14$ seconds. At this point, it has expanded to some radius x. Now, at time $t = 14$, we can express the side lengths of the triangle as $AB = 12, BC = 4 + x, AC = 8 + x$. By the Pythagorean Theorem, we have

$$12^2 + (4 + x)^2 = (8 + x)^2$$

$$\implies 144 + 8x + 16 = 16x + 64$$

$$\implies x = 12$$

   Thus, the length of BC is $4 + x = 4 + 12 = \boxed{16}$ units.

3. A robot has 5 rules under which you are allowed to talk to it:

   (a) You have an interesting question for it.

   (b) It has an interesting question for you.

   (c) You are talking about DMM problems.

   (d) You may ask to add a rule.

   (e) You may ask it about Putnam questions

   Each time you ask to add a rule, a new rule gets added. Every time you and the robot talk, the robot writes down a letter based on the type of conversation. Each day for three days you and the robot talk once. We'll call the potential added rules (f) and (g). As a result, the robot has written down a three-letter string. How many different possibilities are there for this string?

   For example, if we talked about an interesting question you have, and then about adding a new rule, and then about the new rule, the string *adf* would be written down.

   Proposed by: Leo Xu

   Solution: We do casework on the number of rules added.

   If no rules are added, it means we have never asked to add a rule, which has $4^3 = 64$ possibilities for the conversation string for this case.

   If one rule is added, then we have asked to add a rule once. If d is marked for the first day, then there are $5^2 = 25$ possibilities (since we can use the new rule). If d is marked for the second day, then there are $4 \cdot 5 = 20$ possibilities, and if d is marked for the third day, there are $4^2 = 16$ possibilities, for a total of $25 + 20 + 16 = 61$ possibilities for this case.

   If two rules are added, then we have asked to add a rule twice. If d is marked for the first and second day, there are 6 possibilities (since we can use either of the two new rules). If d is marked for the first and third day, there are 5 possibilities, and if d is marked for the second and third day, there are 4 possibilities, for a total of $6 + 5 + 4 = 15$ possibilities for this case.

   Finally, if three rules are added, then we have asked to add a rule thrice, for which there is only 1 possibility.

   Summing gives us $64 + 61 + 15 + 1 = \boxed{141}$ possibilities.

4. Let rectangle $ABCD$ have side lengths $AB = 8$ and $BC = 12$. Point P lies outside rectangle $ABCD$ such that segment $\overline{PA}$ bisects $\overline{BC}$ at M and $PM = 6$. If the value of $PB * PC$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

   Proposed by: Akshat Basannavar

   Solution: Since M is the midpoint of line segment BC, $BM = MC = 6$. Combined with the fact that $PM = 6$, it follows that B, C, P all lie on a circle with center M and radius 6. In particular, BC is the diameter of this circle, so BP, PC are perpendicular by the inscribed angle theorem. We finish by equating area formulas: namely, $2[BPC] = PB \cdot PC = BC \cdot PH$, where H is the point from dropping the altitude from P to BC. Using similar right triangles ABM and PHM, we find $PH = \frac{24}{5}$. Hence, our final answer is $12 \cdot \frac{24}{5} = \frac{288}{5} \implies 288 + 5 = \boxed{293}$.

5. The number 2025 has the interesting property that it is divisible by its sum of digits. This can be generalized by considering the sum of digits in different bases (assuming digits can be greater than 9). As an example, the sum of digits for 2025 in base 100 is

$$20 + 25 = 45$$

   because $2025 = 20 \cdot 100 + 25$. Find the sum of all positive integers n such that n is divisible by its sum of digits in every base $b \geq 2$.

   Proposed by: Erick Jiang

   Solution: Consider a fixed value of n. If $b > n$, then n in base b is just n_b, so we limit our focus to $b \leq n$. So, the bases of interest for a fixed n are 2 to $n - 1$. As a result, we vacuously have that $n = 1, 2$ work.

   For any even integer $2k > 2$, its sum of digits in bases $k + 1, k + 2, \ldots, 2k$ are $k, k - 1, \ldots, 1$ respectively. In particular, since

$$2k = 2(k - 1) + 2$$

   we need $k - 1$ to divide 2, which only holds for $k = 2, 3$. It is quick to check that both $n = 4, 6$ are solutions.

   Now consider an odd integer $2k + 1 > 1$. Its sum of digits in base $k + 2$ is k, so we would need k to divide $2k + 1$. This can only happen if $k = 1$, but $n = 3$ fails in base 2.

   Hence, our final answer is

$$1 + 2 + 4 + 6 = \boxed{13}.$$

6. Let w be a complex number such that $a = 26$ is the smallest positive integer such that $w^a = 1$. Find the number of 26-tuples of 0s and 1s, $(s_1, s_2, ..., s_{26})$, such that $\sum_{n=1}^{26} s_n w^n = 0$.

   Proposed by: Akshat Basannavar

   Solution: Recall that the only solution to $\sum_{n=1}^{p=13} s_n w^n = 0$, where s_n are all integers is precisely when the s_n are all equal. (In fact, this holds for all primes p). Extrapolating to the original problem, note that $w^{t+13} = -w^t$. Hence, we can frame the problem as finding solutions to $\sum_{n=1}^{p=13}(s_n - s_{n+13})w^n = 0$. It suffices to do casework on $s_n - s_{n+13}$. Clearly, this value is either $-1, 0$, or 1. If it is 0, we have $s_i = s_{i+13}$ for all $0 \leq i < 13$, so we can fix these 13 s_i one of two ways and produce a unique combination, giving us 2^{13} possibilities. If the value is -1 or 1, it is easy to see s_i for all $0 \leq i < 26$ has only one choice for each case. The final answer is thus $2^{13} + 2 = \boxed{8194}$.

7. Consider the ordered set of numbers $\{1, 2, 3, ..., n\}$. Let s_i, where $1 \leq i \leq n-1$ be the operation at position i which swaps the number at position i with the number at position $i+1$. Any sequence of s_i's is called a word. For example, let $n = 5$. The word (s_2) turns $\{1, 2, 3, 4, 5\}$ into $\{1, 3, 2, 4, 5\}$, and (s_3, s_4, s_3) turns $\{1, 2, 3, 4, 5\}$ into $\{1, 2, 5, 4, 3\}$. The *length* of a permutation is the number of operations in the shortest possible word which results in that permutation. Let $n = 100$. Over all permutations, what is the maximum *length* of a permutation?

   Proposed by: Havish Shirumalla

   Solution: We first exhibit a permutation whose length is $\binom{n}{2}$. Consider the word

$$w \;=\; (s_1)(s_2s_1)(s_3s_2s_1)\cdots(s_{n-1}s_{n-2}\cdots s_1),$$

   a concatenation of k adjacent transpositions $s_k, s_{k-1}, \ldots, s_1$ for each $1 \leq k \leq n-1$. We claim that w carries $(1, 2, \ldots, n)$ to the reversed permutation $(n, n-1, \ldots, 2, 1)$. We can argue inductively on k. After applying the first block (s_1), the element 2 moves from position 2 to position 1, giving $(2, 1, 3, \ldots, n)$. Assume that after applying the product

$$(s_1)(s_2s_1)\cdots(s_{k-1}\cdots s_1)$$

   the first k positions are occupied by $k, k-1, \ldots, 2, 1$ in that order. The next block $(s_ks_{k-1}\cdots s_1)$ moves the element $k+1$ successively one step to the left, from position $k+1$ to position k, then to $k-1$, and so on until it reaches position 1. Thus after the k-th block the first $k+1$ positions are occupied by $k+1, k, \ldots, 2, 1$. After all $n-1$ blocks we obtain $(n, n-1, \ldots, 2, 1)$ as claimed.

   The length of w is the total number of adjacent transpositions appearing in it, namely

$$1 + 2 + \cdots + (n-1) \;=\; \binom{n}{2}.$$

   Hence the reversed permutation has length at most $\binom{n}{2}$. To prove that this is best possible, we use inversions. For a permutation π of $\{1, \ldots, n\}$, an inversion is a pair (i, j) with $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$. Let $I(\pi)$ denote the number of inversions of π. A single adjacent transposition s_k, swapping $\pi(k)$ and $\pi(k+1)$, can change only the pair $(k, k+1)$: if

$\pi(k) < \pi(k+1)$ then $(k, k+1)$ was not an inversion and becomes one, so $I(\pi)$ increases by 1; if $\pi(k) > \pi(k+1)$ then $(k, k+1)$ was an inversion and ceases to be one, so $I(\pi)$ decreases by 1. Thus each adjacent transposition changes $I(\pi)$ by at most 1 in absolute value.

The identity permutation has $I(\text{id}) = 0$. If a word of length L takes the identity to a permutation π, then after L adjacent transpositions the inversion count can be at most L, so $I(\pi) \leq L$. In particular, any word producing a given permutation π has length at least $I(\pi)$.

The reversed permutation $(n, n-1, \ldots, 1)$ has every pair (i, j) with $i < j$ as an inversion, so

$$I(n, n-1, \ldots, 1) = \binom{n}{2}.$$

Therefore any word yielding the reversed permutation has length at least $\binom{n}{2}$. Combined with the explicit word w above of length $\binom{n}{2}$, this shows that $\binom{n}{2}$ is the maximum possible length of a permutation of $\{1, \ldots, n\}$.

For $n = 100$ this maximum is

$$\binom{100}{2} = \frac{100 \cdot 99}{2} = \boxed{4950},$$

8. Consider polynomial $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 11$, with roots r_1, r_2, r_3, r_4. For each permutation σ of the set $\{1, 2, 3, 4\}$, define

$$U(\sigma) = r_{\sigma(1)}r_{\sigma(2)} + r_{\sigma(3)}r_{\sigma(4)}.$$

If a permutation σ is chosen uniformly at random, the expected value of $U(\sigma)^2$ can be expressed as a common fraction $\frac{m}{n}$ in lowest terms. Find $m + n$.

Proposed by: Havish Shirumalla

Solution: Write the expected value A as the average

$$A = \frac{1}{4!} \sum_{\text{permutations } \sigma} U(\sigma)^2.$$

Note that there are only three distinct values of $U(\sigma)$ which each appear 8 times. We label them

$$U_1 = r_1r_2 + r_3r_4, \quad U_2 = r_1r_3 + r_2r_4, \quad U_3 = r_1r_4 + r_2r_3,$$

and it follows that

$$A = \frac{1}{3}(U_1^2 + U_2^2 + U_3^2).$$

Then, write $s_1 = \sum_i r_i$, $s_2 = \sum_{i<j} r_ir_j$, $s_3 = \sum_{i<j<k} r_ir_jr_k$, and $s_4 = r_1r_2r_3r_4$, all given by Vieta's formulas. We note that

$$U_1^2 + U_2^2 + U_3^2 = (U_1 + U_2 + U_3)^2 - 2\sum_{i<j} U_iU_j,$$

and one verifies that $(U_1+U_2+U_3)^2 = s_2^2$. Writing $2\sum_{i<j} U_iU_j$ in terms of the s_i's is a little harder. Expanding, $U_1U_2+U_1U_3+U_2U_3$, we see that for each ordered triple $i<j<k$, the terms $r_i^2r_jr_k$, $r_ir_j^2r_k$, and $r_ir_jr_k^2$ all appear, so

$$U_1U_2+U_1U_3+U_2U_3 = \sum_{i<j<k}(r_i+r_j+r_k)r_ir_jr_k.$$

Then, note that

$$s_1s_3 = \sum_{i<j<k}(r_i+r_j+r_k+r_l)r_ir_jr_k,$$

where r_l is the omitted root. This is almost our desired sum, but it overcounts 4 copies of $r_1r_2r_3r_4 = s_4$ for each choice of omitted r_l, giving the formula

$$U_1U_2+U_1U_3+U_2U_3 = s_1s_3 - 4s_4.$$

Therefore, $A = \frac{1}{3}(U_1^2+U_2^2+U_3^2) = \frac{1}{3}[s_2^2 - 2(s_1s_3-4s_4)] = \frac{1}{3}[(-7)^2 - 2((-2)\cdot 8 - 4\cdot 11)] = \frac{169}{3}$. Thus, our answer is $169+3 = \boxed{172}$.

9. Let X be a set of 12 distinct real numbers. The set X_c is obtained by adding a real number $c \geq 0$ to each element in X (formally, $X_c = \{x+c : x \in X\}$).

For each $c \geq 0$ such that X and X_c have no elements in common (formally, $X \cap X_c = \emptyset$), define $Y = X \cup X_c$. Sort Y in ascending order. Define binary string S_c such that for $1 \leq k \leq 24$, the kth digit of S_c is 1 if the kth element of Y is in X_c, and 0 otherwise. Let S_X be the set of all such strings S_c.

Over all possibilities of X, what is the maximum number of distinct strings in S_X?

Proposed by: Akshat Basannavar

Solution: Consider placing all elements of X on the real axis. Geometrically, adding c to each element in X is the same as shifting X to the right by c units and producing X_c. Consider what happens as we slowly increase c. Every time a point in X_c moves past a point in X, S_c changes. Furthermore, there are exactly $\binom{12}{2}$ of these changes, because for each distinct pair (a,b) for $a,b \in X$, $a+c$ "shifts" past b exactly once. Hence, we should have $\binom{12}{2}+1 = \boxed{67}$ as an upper bound on the total amount of S_c.

It then suffices to construct an X where there are indeed 67 possible S_c. Indeed, $X = 1, 2, 4, ..., 2^{11}$ is an example of such a set, so we are done.

10. Let I be the intersection of the angle bisectors of triangle ABC and let ray $\overline{AI}$ meet the circle defined by the points A, B, C at D. Denote the feet of the perpendiculars from I to lines $\overline{BD}$ and $\overline{CD}$ by E and F, respectively. If $IE = 2$, $IF = 3$, and $AD = 10$, the greatest possible length of EF^2 can be expressed as $a + b\sqrt{c}$, where a, b, c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.

Proposed by: Nikhil Pesaladinne

Solution: Let the side lengths be $a = BC,\ b = CA,\ c = AB$ and let R be the circumradius.

We first show that $DB = DC = DI$. Since D lies on the internal bisector of $\angle A$,

$$\angle BID = 180^\circ - \angle BIA = 180^\circ - \left(90^\circ + \frac{C}{2}\right) = 90^\circ - \frac{C}{2}.$$

Also $\angle DBI = \angle DBA - \angle IBA = \angle DCA - \frac{B}{2}$. But $\angle DCA = \frac{1}{2}\widehat{DA} = 90^\circ - \frac{C}{2}$ (because D is the midpoint of arc BC containing A). Hence $\angle DBI = \angle BID$, so $DB = DI$. By symmetry, $\angle ICD = \angle CID = 90^\circ - \frac{B}{2}$, giving $DC = DI$. Denote this common length by x: $x = DB = DC = DI$.

Since $\angle IDE = \angle ADB = \angle ACB = C$ and $\angle IDF = \angle ADC = \angle ABC = B$,

$$IE = ID\sin C = x\sin C = \frac{x\,c}{2R}, \qquad IF = ID\sin B = x\sin B = \frac{x\,b}{2R}.$$

With $IE = 2,\ IF = 3$ this gives

$$xc = 4R, \qquad xb = 6R$$

Ptolemy yields

$$(DA)(BC) = (DB)(AC) + (DC)(AB) \implies a \cdot AD = x(b + c).$$

Since $AD = 10$, substitute $xb = 6R,\ xc = 4R$:

$$10a = x(b + c) = 6R + 4R = 10R \implies R = a.$$

By the Law of Sines, $a = 2R\sin A$, so $R = a$ forces $\sin A = \dfrac{a}{2R} = \frac{1}{2}$. Hence $\angle A \in \{30^\circ, 150^\circ\}$. Because $IE \perp BD$ and $IF \perp CD$, the angle between IE and IF equals the angle between BD and CD, i.e.

$$\angle EIF = \angle BDC = 180^\circ - \angle A.$$

By the Law of Cosines in $\triangle EIF$,

$$EF^2 = IE^2 + IF^2 - 2(IE)(IF)\cos\angle EIF = 4 + 9 - 12\cos(180^\circ - A) = 13 + 12\cos A.$$

Thus EF^2 is maximized when $\cos A$ is maximized, i.e. at $A = 30^\circ$. Therefore

$$EF^2_{\max} = 13 + 12\cos 30^\circ = 13 + 6\sqrt{3}$$

Thus, our answer is $13 + 6 + 3 = \boxed{22}$.